

COMMUTATIVE ALGEBRAIC GROUPS AND p -ADIC LINEAR FORMS

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ABSTRACT. Let G be a commutative algebraic group defined over a number field K that is disjoint over K to \mathbb{G}_a and satisfies the condition of semistability. Consider a linear form l on the Lie algebra of G with algebraic coefficients and an algebraic point u in a p -adic neighbourhood of the origin with the condition that l does not vanish at u . We give a lower bound for the p -adic absolute value of $l(u)$ which depends up to an effectively computable constant only on the height of the linear form, the height of the point u and p .

1. INTRODUCTION

The theory of Diophantine approximation is one of the most interesting problems in number theory in which the theory of linear forms plays a central role. In 1966 Baker made a breakthrough by proving a very deep result on effective lower bounds for linear forms in logarithms of algebraic numbers (see the series of papers [1]). This result was refined by Baker and Wüstholz (see [2]). After Wüstholz proved a brilliant theorem which is called the analytic subgroup theorem (see [3] or [23]), the problem of linear forms could be considered in higher dimensions. In the literature one can find generalizations in terms of algebraic groups and the most general results so far are due to Hirata-Kohno (see [13]) and Gaudron (see [12]).

It is natural to consider p -adic analogues of such problems. The theory of p -adic linear forms plays indeed an important and fundamental role in number theory. It has been applied to many questions, in particular it was successfully used to solve completely a large number of Diophantine problems of different shape. One of the interests comes from the problem of finding lower bounds for linear forms in p -adic logarithm functions evaluated at algebraic points. Unlike in the complex case, the p -adic logarithm function is only defined locally. It is therefore more natural to study upper bounds for the p -adic valuation of expressions $\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1$ where $\alpha_1, \dots, \alpha_n$ are algebraic numbers such that they are multiplicatively independent and

2010 *Mathematics Subject Classification.* Primary 11G99; Secondary 14L10, 11J86.

Key words and phrases. commutative algebraic groups, linear forms, effective results, heights.

b_1, \dots, b_n are rational integers, not all zero. Such problems have been investigated by many authors (see e.g. [8]) and the most outstanding results to date are due to Yu (see [26, 27, 28, 29]). In 1998 he formulated and proved a p -adic analogue of the Baker and Wüstholz theorem and afterwards in a series of papers he improved the bounds. The results of Yu were used by Stewart and himself to deal with the *abc*-conjecture (see [20]). In particular, Stewart and Yu in 2001 showed that there is an effectively computable positive number c such that for all coprime positive integers x, y and $z > 2$ with $x + y = z$ one has

$$z < \exp \left(cN^{1/3}(\log N)^3 \right)$$

where N is the product of all the distinct prime divisors of xyz . Furthermore, with the recent refinements of Yu in [29] it is possible to solve completely the generalization of a problem of Erdős to Lucas and Lehmer numbers; the original conjecture of Erdős from 1965 states that $P(2^n - 1)/n \rightarrow \infty$ as $n \rightarrow \infty$, where $P(m)$ denotes the greatest prime divisor of m for integers $m > 1$.

The generalizations to linear forms in p -adic elliptic logarithms were solved by Rémond and Urfels (see [18]), and refined by Hirata-Kohno and Takada (see [14]). For higher dimensions in the p -adic setting, the best results up to date are due to Bertrand and Flicker. They stated some results concerning simple abelian varieties or abelian varieties of CM-type (see [4] and [10]). Flicker also obtained a lower bound for linear forms on general abelian varieties but the bound is ineffective (see [11]).

The goal of this paper is to generalize the result on p -adic linear forms when evaluating at an algebraic point from a commutative algebraic group of positive dimension satisfying a technical condition and the condition of semistability. To describe the main theorem, let K be a number field and G a commutative algebraic group such that G and the additive group \mathbb{G}_a are disjoint over K (see Section 3.2 for the definition of this notion). There are many commutative algebraic groups satisfying this property, for example the direct product of any finite copies of the multiplicative group \mathbb{G}_m or any abelian variety. More generally we prove that every semi-abelian variety also satisfies the property.

Let p be a prime number and consider embeddings $K \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Denote by v the p -adic valuation which is the restriction of the p -adic valuation on \mathbb{C}_p to K and K_v the completion of K with respect to v . We embed G into the projective space \mathbb{P}_K^N for some positive integer N and let $\text{Lie}(G)$ denote the Lie algebra of G . Fixing a choice of basis for the vector space $\text{Lie}(G)$

one can identify $\mathrm{Lie}(G)$ with the vector space K^n ; here n is the dimension of G . We get the normalized analytic function of the exponential map of $G(K_v)$ (with respect to the basis) consisting of N functions analytic on a certain neighbourhood of 0 in K_v^n . Let W be the hyperplane in K^n defined over K by the linear form

$$l(Z_1, \dots, Z_n) = \beta_1 Z_1 + \dots + \beta_n Z_n,$$

where β_1, \dots, β_n are elements, not all zero, in K . Let u be an element in the above neighbourhood such that its image in the p -adic Lie group $G(K_v)$ is an algebraic point γ in $G(K)$. The problem we consider is to give a lower bound for $|l(u)|_p$ when $l(u)$ is non-zero, here as usual we denote by $|\cdot|_p$ the p -adic absolute value on \mathbb{C}_p . The purpose of this paper is to solve the problem in the case when (G, W) is semistable over $\overline{\mathbb{Q}}$. Here we use the condition of semistability introduced in [3] over the algebraic closure $\overline{\mathbb{Q}}$ since it concerns field extensions of the ground field K . Our lower bound consists of two parts; the first one consists of effectively computable constants depending only on the group G , the field K and the choice of basis for the Lie algebra of G , and the second one is the product of the absolute logarithmic (Weil) height of the linear form l , of the algebraic point γ and of the prime number p .

The method used in this paper to solve the problem can certainly be applied to get new results in transcendence theory. We leave this as a topic for a forthcoming paper.

In Section 2 we shall state the new result in detail. In Section 3 we collect some preliminary results including a Schwarz lemma in the p -adic domain, simple facts on disjointness and semistability, on heights, on the analytic representation of the exponential map and a fact about the order of vanishing of analytic functions. In Section 4 we shall give the proof of the main result of Section 2. The proof starts by embedding G into some projective space; this involves a choice which we fix for the rest of the paper. We also choose a basis for the hyperplane. Then we work out the standard program in transcendence theory: we construct an auxiliary function with bounded height and with high order vanishing at certain points. Using the Schwarz lemma we can extrapolate and derive an upper bound. Liouville's inequality from Diophantine approximation gives a lower bound provided that we have non-vanishing. Algebraic considerations (namely multiplicity estimates) give the non-vanishing. Finally, comparing upper and lower bound gives the desired result by an appropriate choice of the parameters.

2. NEW RESULT

As was mentioned above the p -adic theory of logarithmic forms has already been developed systematically with nice applications in number theory. It is therefore natural and clearly motivated to generalize the problem to the case of higher dimensions. There are several results in this direction due to Rémond, Urfels, Hirata-Kohno, Takada, Flicker, Bertrand and others. However, the results only deal with elliptic curves or abelian varieties. We shall give here a new generalization to a class of commutative algebraic groups.

Let K be a number field over \mathbb{Q} and let \mathcal{O}_K be the ring of algebraic integers of K . We choose an embedding $K \hookrightarrow \overline{\mathbb{Q}}$. Let p be a prime number in \mathbb{Z} . We denote by \mathbb{Q}_p the field of p -adic numbers and \mathbb{C}_p the completion of the algebraic closure of \mathbb{Q}_p . We get the embedding $\sigma : K \hookrightarrow \mathbb{C}_p$ defined by the composition of the embeddings $K \hookrightarrow \overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. We therefore identify each element $x \in K$ with $\sigma(x) \in \mathbb{C}_p$. Let v be the valuation on K given by

$$v(x) := -\frac{\log |x|_p}{\log p}, \quad \forall x \in K.$$

Denote by K_v the completion of K with respect to v . By completing the algebraic closure we get $K \hookrightarrow K_v \hookrightarrow \mathbb{C}_p$, which preserves the absolute values. Let G be a commutative algebraic group defined over K of dimension $n \geq 1$. According to [19], see also [9] where explicit embeddings are constructed using exponential- and Theta-functions, G can be embedded into some projective space \mathbb{P}^N . Let $L : \{1, \dots, n\} \rightarrow \text{Lie}(G)$ be a basis, $f_L = (f_1, \dots, f_N)$ the normalized analytic function of the exponential map of $G(K_v)$ with respect to L and Exp the map as defined in Section 3.5. We know that f_1, \dots, f_N are analytic on an open disk Λ_v of K_v^n (see again Section 3.5). Let W be the hyperplane in K^n defined over \mathcal{O}_K by the linear form

$$l(Z_1, \dots, Z_n) = \beta_1 Z_1 + \dots + \beta_n Z_n,$$

where β_1, \dots, β_n are elements, not all zero, in \mathcal{O}_K . Let u be an element in Λ_v such that $\gamma := \text{Exp}(u)$ is an algebraic point in $G(K)$. Let B and H be fixed numbers such that

$$B \geq \max_{i=1, \dots, n} \{3, H(\beta_i)\}, \quad H \geq \max\{3, H(\gamma)\}.$$

Put $b = \log B$ and $h = \log H$. If $u = (u_1, \dots, u_n)$ is not contained in $W_v := W \otimes_K K_v$, i.e. $l(u) = \beta_1 u_1 + \dots + \beta_n u_n \neq 0$, then a natural question is “*What can we say about lower bounds for $|l(u)|_p$?*”. Below we give an answer to this question in the case when G, \mathbb{G}_a are disjoint over K (for

example, G is semi-abelian, see Lemma 3.5) and (G, W) is semistable over $\overline{\mathbb{Q}}$. Let δ_L be the denominator of L which is defined in Section 3.5 and let $B^n(r_p|\delta_L|_p)$ denote the set $\{x = (x_1, \dots, x_n) \in \mathbb{C}_p^n; |x_i|_p < r_p|\delta_L|_p \text{ for } i = 1, \dots, n\}$, where $r_p := p^{-1/(p-1)}$. Then we have the following:

Theorem 2.1. Let K be a number field and G a commutative algebraic group of dimension $n \geq 1$ defined over K such that G and \mathbb{G}_a are disjoint over K and such that (G, W) is semistable over $\overline{\mathbb{Q}}$. There is a positive number ω_L depending on L and there exist effectively computable positive real constants c_0 and c_1 independent of b, h and p with the following property:
 1. If $u \in \Lambda_v \cap B^n(r_p|\delta_L|_p)$ such that $\text{Exp}(u)$ is an algebraic point in $G(K)$ then $l(u) = 0$ or

$$\log |l(u)|_p > -c_0 \omega_L^{n+3} b h^n (\log b + \log h)^{n+3} \log p.$$

2. If $u \in \Lambda_v$ such that $\text{Exp}(u)$ is an algebraic point in $G(K)$ then we put

$$n(u) := \max \left\{ 0, \left\lceil \frac{1}{p-1} - v(u) \right\rceil + 1 \right\}$$

and either $l(u) = 0$ or we get the lower bound

$$\log |l(u)|_p > -c_1 \omega_L^{n+3} b h^n (\log b + \log h + 2n(u) \log p)^{n+3} \log p.$$

Throughout the paper constants do not depend on b, h and p . We write $A \ll B$ (resp. $A \gg B$) if there is an effectively computable positive constant c such that $A \leq cB$ (resp. $A \geq cB$).

We remark that although in the above theorem we only consider the case $\beta_1, \dots, \beta_n \in \mathcal{O}_K$ the theorem is still true for $\beta_1, \dots, \beta_n \in K$. To see this, let δ_i be the denominator of β_i for $i = 1, \dots, n$ and δ the least common multiple of $\delta_1, \dots, \delta_n$. Put $\beta'_i := \delta \beta_i$ for $i = 1, \dots, n$ and $l' = \delta l$. Then $\beta'_1, \dots, \beta'_n \in \mathcal{O}_K$ and $|l(u)|_p = |\delta^{-1}|_p |l'(u)|_p \geq |l'(u)|_p$. Using Lemma 3.8 we get $\log \delta \leq \log(\delta_1 \cdots \delta_n) = \log \delta_1 + \cdots + \log \delta_n \ll b$, and this gives $h(\beta'_i) = h(\delta \beta_i) \ll b$ for all $i = 1, \dots, n$. Hence the statement follows by applying Theorem 2.1 to the linear form l' and from the inequality $\log |l(u)|_p \geq \log |l'(u)|_p$.

We also remark that it would be nice to remove the technical assumptions concerning disjointness and semistability in the statement. This clearly needs some further efforts. Since the paper is already quite long, we leave this for future work.

3. BACKGROUND AND PRELIMINARIES

In this section we discuss some basic background material which we need later for the proof of the main theorem.

3.1. Some p -adic analysis. The main result of this section is a Schwarz lemma in the p -adic domain which will be given in Proposition 3.3 below. For any subfield F of \mathbb{C}_p and for any $R \geq 0$ we set $B_F(R) := \{x \in F; |x|_p < R\}$ and $\overline{B}_F(R) := \{x \in F; |x|_p \leq R\}$. From now on, we will skip the subscript F when $F = \mathbb{C}_p$. Let $f(x) = \sum_n a_n x^n$ be an analytic function on $\overline{B}(r)$ with $r > 0$. We define

$$|f|_r := \sup_n |a_n|_p r^n = \max_n |a_n|_p r^n.$$

We start with the remark that the function $z - a$ satisfies $|z - a|_r = r$ for $r > 0$ and for $a \in \overline{B}_F(r)$. Indeed, by definition we have $|z - a|_r = \max\{|a|_p, r\} = r$.

Lemma 3.1. Let f be an analytic function on $\overline{B}_F(r)$ with $r > 0$, and s, t real numbers such that $0 < s \leq t \leq r$. If f has k zeros in the disk $\overline{B}_F(s)$ then

$$|f|_s \leq \left(\frac{s}{t}\right)^k |f|_t.$$

Proof. The statement is trivially true if $f \equiv 0$. Otherwise, the Weierstrass preparation theorem (see Theorem 2.14 in [16]) says that one can write $f = P \cdot g$ with $P(z) = (z - a_1) \cdots (z - a_k)$ for $a_1, \dots, a_k \in \overline{B}_F(s)$ and with a certain analytic function g on $\overline{B}_F(r)$. By the remark above we get

$$|P|_s = |z - a_1|_s \cdots |z - a_k|_s = s^k$$

and similarly for $|P|_t$. Hence

$$|f|_s = s^k |g|_s \leq s^k |g|_t = \left(\frac{s}{t}\right)^k t^k |g|_t = \left(\frac{s}{t}\right)^k |f|_t,$$

and this ends the proof. \square

Lemma 3.2. Let f be an analytic function on $\overline{B}(r)$ with $r > 0$ and s, t be real numbers such that $0 < s \leq t \leq r$. Let m be the number of zeros (counted with multiplicities) of f in $B(t)$ then

$$|f|_t \leq \left(\frac{t}{s}\right)^m |f|_s.$$

Proof. The statement is trivial if $f \equiv 0$ or $s = t$. Otherwise, let b_1, \dots, b_m be the zeros of f in $B(t)$ (counted with multiplicities) and let t' be a fixed real number such that

$$\max\{|b_1|_p, \dots, |b_m|_p\} < t' < t.$$

Let l be the number of zeros (counted with multiplicities) of f in $\overline{B}(s)$. Without loss of generality, we may assume that b_1, \dots, b_l are the l zeros of f in $\overline{B}(s)$. By the Weierstrass preparation theorem (see Theorem 2.14 in [16]) there are $\alpha_1, \alpha_2 \in \mathbb{C}_p$ and functions g_1, g_2 such that g_1 is analytic on

$\overline{B}(s)$ and g_2 is analytic on $\overline{B}(t)$, $g_1(0) = g_2(0) = 1$, $|g_1|_s = |g_2|_r = 1$, and $f(z) = \alpha_1(z - b_1) \cdots (z - b_l)g_1 = \alpha_2(z - b_1) \cdots (z - b_m)g_2$. Combining this with the above remark we get

$$|f|_s = |\alpha_1|_s |z - b_1|_s \cdots |z - b_l|_s |g_1|_s = |\alpha_1|_p s^l$$

and

$$|f|_{t'} = |\alpha_2|_{t'} |z - b_1|_{t'} \cdots |z - b_m|_{t'} |g_2|_{t'} = |\alpha_2|_p t'^m.$$

Hence

$$|f|_t = \lim_{t' \rightarrow t} |f|_{t'} = |\alpha_2|_p t^m.$$

On the other hand, since $g_1(0) = g_2(0) = 1$ it follows that

$$f(0) = \alpha_1(-1)^l b_1 \cdots b_l = \alpha_2(-1)^m b_1 \cdots b_m$$

which leads to

$$|\alpha_1|_p = |\alpha_2|_p |b_{l+1} \cdots b_m|_p.$$

This shows that

$$\frac{|f|_t}{|f|_s} = \frac{|\alpha_2|_p t^m}{|\alpha_1|_p s^l} = \frac{t^m}{s^m} \frac{s^{m-l}}{|b_{l+1} \cdots b_m|_p}.$$

Since $b_{l+1}, \dots, b_m \in B(t) \setminus \overline{B}(s)$ it follows that

$$|b_{l+1} \cdots b_m|_p \geq s^{m-l}.$$

Hence

$$\frac{|f|_t}{|f|_s} \leq \frac{t^m}{s^m},$$

and this is equivalent to the inequality

$$|f|_t \leq \left(\frac{t}{s}\right)^m |f|_s$$

which proves the statement. \square

We are now able to prove the following proposition which is called Schwarz lemma.

Proposition 3.3. Let $t \geq s$ be positive real numbers, f an analytic function on $\overline{B}_F(t)$ and Γ a finite subset of $\overline{B}_F(s)$ of cardinality $l \geq 2$. We define

$$\delta := \inf\{|\gamma - \gamma'|_p; \gamma, \gamma' \in \Gamma, \gamma \neq \gamma'\}$$

and

$$\mu := \sup\{|f^{(n)}(\gamma)|_p; n = 0, \dots, k-1, \gamma \in \Gamma\}$$

with a positive integer k and with $f^{(n)}$ the n -th derivative of f . Assume that $|\delta|_p \leq 1$ then

$$|f|_s \leq \max \left\{ \left(\frac{s}{t}\right)^{kl} |f|_t, \mu \left(\frac{s}{\delta}\right)^{kl-1} r_p^{-(k-1)} \right\}.$$

Proof. The proposition is trivially true if $f \equiv 0$ and therefore we may assume that f is non-zero. If f has at least kl zeros in the disc $\overline{B}(s)$ then Lemma 3.1 gives

$$|f|_s \leq \left(\frac{s}{t}\right)^{kl} |f|_t.$$

Otherwise f has at most $kl - 1$ zeros in the disc $\overline{B}(s)$. By the definition of δ , the sets $B(\gamma, \delta), \gamma \in \Gamma$ are disjoint. In fact, suppose that there exist two distinct elements γ_1 and γ_2 in Γ such that there is $x \in B(\gamma_1, \delta) \cap B(\gamma_2, \delta)$. Then this leads to the following contradiction

$$|\gamma_1 - \gamma_2|_p \leq \max\{|x - \gamma_1|_p, |x - \gamma_2|_p\} < \delta.$$

Furthermore these l sets $B(\gamma, \delta), \gamma \in \Gamma$, are subsets of $\overline{B}(s)$ since Γ is contained in $\overline{B}_F(s)$, and this shows that there exists an element γ_0 of Γ such that f has at most $k - 1$ zeros in $B(\gamma_0, \delta)$. Since $\gamma_0 \in \overline{B}_F(s)$ it gives $|f(z - \gamma_0)|_r = |f(z)|_r$ for any r such that $s \leq r \leq t$. We may therefore assume without loss of generality that $\gamma_0 = 0$. Let $n(\delta, f)$ be the number of zeros (counted with multiplicities) of f in $B(\delta)$. It is clear that $n(\delta, f) \leq k - 1$, and this shows that

$$|f|_\delta = \sup_{n \leq k-1} \left| \frac{f^{(n)}(0)}{n!} \right|_p \delta^n.$$

On the other hand, it is known that

$$\left| \frac{1}{n!} \right|_p \leq p^{\frac{n-1}{p-1}} = r_p^{-(n-1)} \leq r_p^{-(k-1)}.$$

Combining this with $|\delta|_p \leq 1$, we get

$$|f|_\delta \leq \mu r_p^{-(k-1)}.$$

Finally, since f has at most $kl - 1$ zeros in $\overline{B}(s)$, Lemma 3.2 gives

$$|f|_s \leq \left(\frac{s}{\delta}\right)^{kl-1} |f|_\delta$$

and this shows the proposition. \square

3.2. Semi-abelian varieties. Let G be an algebraic group defined over a field K . It is well-known from Chevalley's theorem that there is a unique short exact sequence of algebraic groups

$$1 \rightarrow H \rightarrow G \rightarrow A \rightarrow 1$$

with H a linear algebraic group and A an abelian variety defined over K . We call G a semi-abelian variety if in the above exact sequence the group H is a torus, i.e. $H_{\overline{K}} \cong (\mathbb{G}_m \otimes \overline{K})^k$ for some $k \geq 0$; here \mathbb{G}_m denotes the multiplicative group. One can show that G is semi-abelian defined over K if

and only if $G_{\overline{K}}$ is semi-abelian defined over \overline{K} . It is known that every semi-abelian variety is commutative (see [25, Proposition 2.3]). We recall the following definition which is given in a paper of Wüstholz and Masser (see [15]): Let G_1, \dots, G_k be algebraic groups defined over K . We say that they are (mutually) disjoint over K if every connected algebraic K -subgroup H of $G := G_1 \times \dots \times G_k$ has the form $H_1 \times \dots \times H_k$ for algebraic K -subgroups H_1, \dots, H_k of G_1, \dots, G_k respectively.

Lemma 3.4. For S semi-abelian $\text{Hom}(S, \mathbb{G}_a) = (0)$.

Proof. Notice that $S(\overline{K})_{\text{tor}}$ is Zariski dense in S and that any homomorphism α maps $S(\overline{K})_{\text{tor}}$ to $\mathbb{G}_a(\overline{K})_{\text{tor}} = (0)$. It follows that $\alpha(S) = (0)$ and this gives $\alpha = 0$. \square

Lemma 3.5. Every semi-abelian variety defined over K and the additive group \mathbb{G}_a are disjoint over K .

Proof. Let \mathcal{H} be an arbitrary algebraic K -subgroup of $\mathcal{G} := \mathbb{G}_a \times G$. By making a base change to \overline{K} we may assume, without loss of generality, that $K = \overline{K}$. We denote by π_a and π the projections of \mathcal{H} on \mathbb{G}_a and on G respectively. Put $H_a := \pi_a(\mathcal{H} \cap (\mathbb{G}_a \times \{e\}))$ and $H := \pi(\mathcal{H} \cap (\{0\} \times G))$. Then H_a is an algebraic K -subgroup of \mathbb{G}_a and H is an algebraic K -subgroup of G . Let P be the image of \mathcal{H} under the projection

$$\mathbb{G}_a \times G \rightarrow (\mathbb{G}_a \times G)/(H_a \times H) \cong (\mathbb{G}_a/H_a) \times (G/H).$$

Define p_a and p the projections of $(\mathbb{G}_a/H_a) \times (G/H)$ onto \mathbb{G}_a/H_a and onto G/H respectively. We show that $P \cong p_a(P)$ and $P \cong p(P)$. For the first isomorphism, since p_a is surjective it is sufficient to show the restriction of p_a to P is injective. In fact, let (x, y) be any element in \mathcal{H} such that $p_a((x, y)(H_a \times H)) = H_a$, and this means that $x \in H_a$. But $H_a = \pi_a(\mathcal{H} \cap (\mathbb{G}_a \times \{e\}))$, it follows that $(x, e) \in \mathcal{H}$. Combining this with $(x, y) \in \mathcal{H}$ we imply that $(0, y) \in \mathcal{H}$. Thus $y = \pi(0, y) \in \pi(\mathcal{H} \cap (\{0\} \times G)) = H$, and this shows that $(x, y) \in H_a \times H$. By the same argument, we also get the second isomorphism.

Since G is semi-abelian G/H is semi-abelian as well. It follows from above that $P \cong p(P)$ is semi-abelian. By Lemma 3.4 we get $\text{Hom}(P, \mathbb{G}_a) = (0)$. Furthermore, it is clear that H_a is either trivial or \mathbb{G}_a hence $p_a(P) \subseteq \mathbb{G}_a$. This says that $p_a \in \text{Hom}(P, \mathbb{G}_a) = (0)$ which gives $P \cong p_a(P) = (0)$ and implies that $\mathcal{H} = H_a \times H$. \square

3.3. Semistability. We recall the following notion which is due to Wüstholz (see [3, Chapter 6]). Let G be an algebraic group defined over a field K and

V a K -linear subspace of the Lie algebra $\text{Lie}(G)$ of G . We associate with (G, V) the index

$$\tau(G, V) := \begin{cases} \frac{\dim V}{\dim G} & \text{if } \dim G > 0, \\ 1 & \text{otherwise.} \end{cases}$$

The pair (G, V) is called semistable (over K) if for any proper quotient $\pi : G \rightarrow H$ defined over K , we have $\tau(G, V) \leq \tau(H, \pi_*(V))$ where $\pi_* : \text{Lie}(G) \rightarrow \text{Lie}(H)$ is the K -linear map induced by π . Let F/K be a field extension. We say that (G, V) is semistable over F if $(G_F, V \otimes_K F)$ is semistable.

3.4. Heights. Let K be a number field of degree d over \mathbb{Q} , and M_K the set of places of K . For a place $v \in M_K$ we write K_v for the completion of K at v and introduce the normalized absolute value $|\cdot|_v$ as follows. If $v \mid p$ we define $|p|_v := p^{-[K_v:\mathbb{Q}_p]}$. If $v \mid \infty$ it corresponds to the embedding τ_v of K into \mathbb{C} , and we define $|x|_v := |\tau_v(x)|^{[K_v:\mathbb{R}]}$ for any $x \in K_v$. One can show that

$$\prod_{v \in M_K} |x|_v = 1, \quad \forall x \in K \setminus \{0\},$$

and this is called the product formula. Let $P \in \mathbb{P}^n(K)$ be a point represented by a homogeneous non-zero vector x with coordinates x_0, \dots, x_n . We set

$$h_K(x) := \sum_{v \in M_K} \max_i \log |x_i|_v.$$

The absolute logarithmic (Weil) height H on $\mathbb{P}^n(\overline{\mathbb{Q}})$ is defined by

$$h(P) := \frac{1}{[K:\mathbb{Q}]} h_K(x)$$

where K is any number field containing P , and the absolute (Weil) height of P is defined by $H(P) := e^{h(P)}$.

Let α be an element in $\overline{\mathbb{Q}}$. We define $h(\alpha)$ as the absolute logarithmic height of the point in $\mathbb{P}^1(K)$ with projective coordinates $1, \alpha$. It is known that $h(\alpha_1 \cdots \alpha_r) \leq h(\alpha_1) + \cdots + h(\alpha_r)$ and $h(\alpha_1 + \cdots + \alpha_r) \leq \log r + h(\alpha_1) + \cdots + h(\alpha_r)$ with $r \geq 1$ and with $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}$. Let $x = (x_1, \dots, x_n)$ be an element in $\mathbb{A}^n(K)$. We define

$$|x|_v := \max_i |x_i|_v, \quad \forall v \in M_K,$$

and

$$h_{\max}(x) := \sum_{v \in M_K} \log |x|_v$$

for $x \neq 0$, otherwise we put $h_{\max}(0) := 0$. It is convenient to introduce the function

$$h_{L^2}(x) := \sum_{v \in M_K} \log |x|_{L^2, v}$$

where

$$|x|_{L^2, v} = \begin{cases} \max_i |x_i|_v & v \text{ non-archimedean} \\ \left(\sum_i \tau_v(x_i)^2 \right)^{\frac{1}{2}} & v \text{ real} \\ \sum_i \tau_v(x_i) \overline{\tau_v(x_i)} & v \text{ complex.} \end{cases}$$

We write $\log^+ t$ for $\max\{0, \log t\}$ for any positive real number t , extended by $\log^+ 0 = 0$. Put

$$H_{\max}^+ := \prod_{v \in M_K} \max\{|x|_v, 1\},$$

$$h_{\max}^+(x) := \log H_{\max}^+(x) = \sum_{v \in M_K} \log^+ |x|_v,$$

and

$$h_{L^2}^+(x) = \sum_{v \in M_K} \log^+ |x|_{L^2, v}.$$

These heights are related by

$$h_{\max} \leq h_{L^2} \leq h_{\max} + \frac{d}{2} \log(n+1)$$

and

$$h_{\max}^+ \leq h_{L^2}^+ \leq h_{\max}^+ + \frac{d}{2} \log(n+1).$$

If we identify each point $x = (x_1, \dots, x_n) \in \mathbb{A}^n(K)$ with the projective point $(1 : x_1 : \dots : x_n)$ then by definition one gets $h_K(x) = h_{\max}^+(x)$.

One can extend the notations given above to polynomials in n variables T_1, \dots, T_n with coefficients in K . In more details, let $P = \sum_i a_i T^i$ be such a polynomial with $i : \{1, \dots, n\} \rightarrow \mathbb{N}^n$ a multi-index and $T^i = T_1^{i(1)} \dots T_n^{i(n)}$. It corresponds to a point $a = (\dots, a_i, \dots)$ in an affine space $\mathbb{A}^N(K)$ and we define

$$|P|_v := |a|_v, \quad |P|_{L^2, v} := |a|_{L^2, v}$$

and the heights of P as 0 for $P = 0$ and for $P \neq 0$ as

$$h_{\max}(P) = \sum_{v \in M_K} \log |P|_v, \quad h_{L^2}(P) = \sum_{v \in M_K} \log |P|_{L^2, v}.$$

We shall also use

$$h_{\max}^+(P) = \sum_{v \in M_K} \log^+ |P|_v, \quad h_{L^2}^+(P) = \sum_{v \in M_K} \log^+ |P|_{L^2, v}.$$

Proposition 3.6 (Siegel's lemma). Let $N > M$ be positive integers and let l_1, \dots, l_M be linear forms in N variables in T_1, \dots, T_N with coefficients in K . Then there exists a non-trivial solution $x = (x_1, \dots, x_N) \in \mathcal{O}_K^N$ for the system of linear equations $l_1(T_1, \dots, T_N) = \dots = l_M(T_1, \dots, T_N) = 0$ such that

$$h_{\max}^+(x) \leq \frac{1}{2} \log |\text{disc}(K)| + M/(N - M) \max_i h_{L^2}(l_i)$$

where $\text{disc}(K)$ denotes the field discriminant of K .

Proof. This is Corollary 11 of [6]. \square

We recall the Liouville's inequality for number fields which is simple but has an important role in the proof of the main theorem below.

Proposition 3.7 (Liouville's inequality). Let K be a number field and let α be a non-zero element in K . Then

$$\log |\alpha|_v \geq -\frac{h(\alpha)}{[K : \mathbb{Q}]}, \quad \forall v \in M_K.$$

Proof. This is [5, Corollary 2.9.2]. \square

For an algebraic number $\alpha \in K$, the denominator δ of α is defined as the smallest positive integer for which the element $\delta\alpha$ is in \mathcal{O}_K . For a polynomial P with coefficients $a_i, i \in I$, in K , we define the denominator $\delta(P)$ of P as the smallest positive integer for which the elements $\delta(P)a_i \in \mathcal{O}_K$ for all $i \in I$. The following lemma gives an inequality between the height and the denominator of an algebraic number.

Lemma 3.8. Let α be an element in K and δ its denominator. One has

$$\log \delta \leq \frac{h(\alpha)}{[K : \mathbb{Q}]}.$$

Proof. For $v \in M_K \setminus M_K^\infty$ let p be the residue characteristic of v . By definition

$$|\alpha|_v = |N_{K_v/\mathbb{Q}_p}(\alpha)|_p^{\frac{1}{[K_v:\mathbb{Q}_p]}} = |N_{K_v/\mathbb{Q}_p}(\alpha)|_p^{\frac{1}{n_v}}$$

with n_v the degree of K_v over \mathbb{Q}_p . Since $N_{K_v/\mathbb{Q}_p}(\alpha)$ is an element in \mathbb{Q}_p and since the value group of \mathbb{Q}_p is \mathbb{Z} , the element

$$m_v := \frac{n_v}{\log p} \max\{\log |\alpha|_v, 0\}$$

is a non-negative integer. Let S be the set $\{(p, v); p \text{ the residue characteristic of } v, v \in M_K \setminus M_K^\infty, |\alpha|_v > 1\}$. One has S is a finite set. We see that

$$\prod_{(p,v) \in S} p^{m_v} \alpha \in \mathcal{O}_K.$$

This shows, by definition of the denominator of α , that

$$\delta \leq \prod_{(p,v) \in S} p^{m_v}$$

and therefore

$$\log \delta \leq \frac{h(\alpha)}{[K : \mathbb{Q}]}.$$

The lemma is proved. \square

3.5. Analytic representation of exponential maps. Let K be a number field and let G be an algebraic group defined over K . We denote by \overline{G} the Zariski closure of G in \mathbb{P}^N . Let U be the open affine subset defined by $\overline{G} \cap \{X_0 \neq 0\}$. We know that the affine algebra $\Gamma(U, \mathcal{O}_{\overline{G}})$ is stable under the action of any element in $\mathfrak{g} = \text{Lie}(G)$ and it is generated by ξ_1, \dots, ξ_N , where

$$\xi_i := \left(\frac{X_i}{X_0} \right) \Big|_U, \quad \forall i = 1, \dots, N$$

(see [23]). We call a map $L : \{1, \dots, n\} \rightarrow \mathfrak{g}$ a basis if $L(1), \dots, L(n)$ is a basis for \mathfrak{g} . With such a basis L , one gets a system of polynomials $P_{i,L(j)}$ in N variables such that

$$L(j)\xi_i = P_{i,L(j)}(\xi_1, \dots, \xi_N), \quad \forall i = 1, \dots, N, \forall j = 1, \dots, n.$$

This means that

$$\mathcal{L}_j := L(j)(\mathcal{O}_K[\xi_1, \dots, \xi_N])$$

is an \mathcal{O}_K -module in $K[\xi_1, \dots, \xi_N]$ for any $j = 1, \dots, n$. Put $\mathcal{L} = \mathcal{L}_1 + \dots + \mathcal{L}_n$ and define

$$\mathcal{I}_L := (\mathcal{O}_K[\xi_1, \dots, \xi_N] : \mathcal{L}) = \{t \in \mathcal{O}_K; t\mathcal{L} \subset \mathcal{O}_K[\xi_1, \dots, \xi_N]\}.$$

Then \mathcal{I}_L is an ideal of \mathcal{O}_K and its norm $N_{K:\mathbb{Q}}(\mathcal{I}_L)$ is an ideal in \mathbb{Z} which has to be principal. It takes the form (δ_L) for some positive integer δ_L . We call δ_L the denominator of L .

Denote by $\partial_1, \dots, \partial_n$ the canonical basis of $\text{Lie}(K_v^n)$ defined as $\partial_i x_j = \delta_{ij}$ for all $i = 1, \dots, n$ and for all $j = 1, \dots, N$, where δ_{ij} are Kronecker's delta and x_i are the coordinate functions of K_v^n . We define the isomorphisms

$$\partial : K_v^n \rightarrow \text{Lie}(K_v^n), \quad x = (x_1, \dots, x_n) \mapsto x_1 \partial_1 + \dots + x_n \partial_n$$

and

$$\iota : \text{Lie}(K_v^n) \rightarrow \text{Lie}(G(K_v)), \quad \iota(\partial_1) = L(1), \dots, \iota(\partial_n) = L(n).$$

We consider now the set $G(K_v)$ of K_v -points of G . It is known that $G(K_v)$ is a Lie group over K_v . By [7, Chapter III, §7], there is a map \exp (which

is called exponential map) defined and locally analytic on an open disk U_v of $\text{Lie}(G(K_v))$. The functions

$$f_i := \xi_i \circ \text{Exp}, \quad i = 1, \dots, N$$

are analytic on $\Lambda_v := (\iota \circ \partial)^{-1}(U_v)$ in K_v^n , where $\text{Exp} = \exp \circ \iota \circ \partial$.

Let $\mathcal{O}_{G(K_v)}, \mathcal{O}_{U_v}, \mathcal{O}_{\partial(\Lambda_v)}$ and \mathcal{O}_{Λ_v} be the sheaves of analytic functions on $G(K_v), U_v, \partial(\Lambda_v)$ and Λ_v , respectively. So we get commutative diagrams

$$\begin{array}{ccccccc} \mathcal{O}_{G(K_v)} & \xrightarrow{\text{exp}^*} & \mathcal{O}_{U_v} & \xrightarrow{\iota^*} & \mathcal{O}_{\partial(\Lambda_v)} & \xrightarrow{\partial^*} & \mathcal{O}_{\Lambda_v} \\ \downarrow L(j) & & & & & & \downarrow \partial_j \\ \mathcal{O}_{G(K_v)} & \xrightarrow{\text{exp}^*} & \mathcal{O}_{U_v} & \xrightarrow{\iota^*} & \mathcal{O}_{\partial(\Lambda_v)} & \xrightarrow{\partial^*} & \mathcal{O}_{\Lambda_v} \end{array}$$

for all $j = 1, \dots, n$. This leads to

$$(\partial_j \circ \text{Exp}^*)(\xi_i) = (\text{Exp}^* \circ L(j))(\xi_i), \quad \forall i = 1, \dots, N,$$

i.e.

$$\partial_j(f_i) = L(j)(\xi_i) \circ \text{Exp} = P_{i,L(j)}(\xi_1, \dots, \xi_N) \circ \text{Exp} = P_{i,L(j)}(f_1, \dots, f_N)$$

for any $i = 1, \dots, N$ and $j = 1, \dots, n$.

The map $f_L = (f_1, \dots, f_N) : \Lambda_v \rightarrow K_v^N$ is called the normalized analytic representation of the exponential map \exp with respect to the basis L . We define

$$d_L := \max_{i,j} \deg P_{i,L(j)}; \quad e_L := v(\delta_L); \quad h_L := \max_{i,j} h(P_{i,L(j)})$$

and

$$\omega_L := \max\{1, e_L\}(h_L + \log \delta_L + \log d_L);$$

here by convention, $\log d_L = 0$ if $d_L = 0$.

We fix the following notations. For $m = (m_1, \dots, m_k) \in \mathbb{N}^k$ with $0 \leq k \leq n$, we write

$$\partial^m := \partial_1^{m_1} \dots \partial_k^{m_k}; \quad L^m := L(1)^{m_1} \dots L(k)^{m_k}; \quad |m| := m_1 + \dots + m_k.$$

Lemma 3.9. Let $L : \{1, \dots, n\} \rightarrow \mathfrak{g}$ be a basis and $P(T_1, \dots, T_N)$ a polynomial in N variables with coefficients in K of total degree $\leq D$. Let T be a non-negative integer and $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ be such that $T = t_1 + \dots + t_n$. There exists a polynomial $P_t \in K[T_1, \dots, T_N]$ such that

$$\partial^t P(f_1, \dots, f_N) = P_t(f_1, \dots, f_N),$$

satisfying

1. $\deg P_t \leq D + T(d_L - 1)$,
2. $\log |P_t|_v \ll \log |P|_v + T(h_L + \log(D + Td_L)), \quad \forall v \in M_K$.

Proof. We shall prove the lemma by induction on $T = |t|$. The lemma is trivially true for $|t| = 0$. Assume that it is true for any $t \in \mathbb{N}^n$ with $|t| = T \geq 0$. We prove it is also true for any $t \in \mathbb{N}^n$ with $|t| = T + 1$. Let $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ be such that $t_1 + \dots + t_n = T + 1$. Without loss of generality, we may assume that $t_1 \geq 1$. Put $\tau = (t_1 - 1, \dots, t_n)$, by induction one gets

$$\partial^\tau P(f_1, \dots, f_N) = P_\tau(f_1, \dots, f_N)$$

with

$$D_\tau := \deg P_\tau \leq D + T d_L$$

and

$$\log |P_\tau|_v \ll \log |P|_v + T(h_L + \log(D + T d_L)).$$

We write

$$P_\tau = \sum_{m_1 + \dots + m_N \leq D_\tau} a(m_1, \dots, m_N) T_1^{m_1} \dots T_N^{m_N} = \sum_m a(m) T_1^{m_1} \dots T_n^{m_n}$$

and

$$P_{i,L(1)} = \sum_{m_{i,1} + \dots + m_{i,N} \leq d_L} a(m_{i,1}, \dots, m_{i,N}) T_1^{m_{i,1}} \dots T_N^{m_{i,N}}$$

with the coefficients $a(m_{i,1}, \dots, m_{i,N}) \in K$ for all $1 \leq i \leq N$. This gives

$$\partial_1 f_i = \sum_{m_{i,1} + \dots + m_{i,N} \leq d_L} a(m_{i,1}, \dots, m_{i,N}) f_1^{m_{i,1}} \dots f_N^{m_{i,N}}, \quad \forall i = 1, \dots, N.$$

Since $\partial^t = \partial_1 \partial_1^{t_1-1} \dots \partial_n^{t_n} = \partial_1 \partial^\tau$ it follows that

$$\begin{aligned} \partial^t P(f_1, \dots, f_N) &= \partial_1 \partial^\tau P(f_1, \dots, f_N) = \partial_1 P_\tau(f_1, \dots, f_N) \\ &= \sum_m a(m) \sum_{i=1}^N m_i \left(\prod_{j \neq i} f_j^{m_j} \right) f_i^{m_i-1} \partial_1 f_i \end{aligned}$$

which is expanded as

$$\sum_m \sum_{i=1}^N \sum_{m_{i,1} + \dots + m_{i,N} \leq d_L} m_i a(m) a(m_{i,1}, \dots, m_{i,N}) \left(\prod_{j \neq i} f_j^{m_j + m_{i,j}} \right) f_i^{m_i + m_{i,i} - 1}.$$

This shows that

$$\partial^t P(f_1, \dots, f_N) = P_t(f_1, \dots, f_N)$$

for a certain polynomial

$$P_t(T_1, \dots, T_N) = \sum_l q(l) T_1^{l_1} \dots T_N^{l_N}$$

with $q(l) = \sum m_i a(m) a(m_{i,1}, \dots, m_{i,N})$; here the sum is taken over the set $\{(m_1, \dots, m_N, i, m_{i,1}, \dots, m_{i,N}); m_j + m_{i,j} = l_j \text{ for } j \neq i \text{ and } m_i + m_{i,i} =$

$l_i + 1, 1 \leq i \leq N, m_{i,1} + \cdots + m_{i,N} \leq d_L, m_1 + \cdots + m_N \leq D_\tau\}$ such that

$$\begin{aligned} \deg P_t &\leq \max_i (m_1 + \cdots + m_N + m_{i,1} + \cdots + m_{i,N} - 1) \\ &\leq D_\tau + d_L - 1 \leq D + T(d_L - 1) + d_L - 1 \\ &\leq D + (T + 1)(d_L - 1). \end{aligned}$$

Furthermore we find that

$$\begin{aligned} |q(l)|_v &\leq \sum m_i |a(m)|_v |a(m_{i,1}, \dots, m_{i,N})|_v \\ &\leq (d_L + 1)^N D_\tau |P_\tau|_v \max_{i,j} |P_{i,L(j)}|_v. \end{aligned}$$

This shows that

$$\begin{aligned} \log |q(l)|_v &\leq N \log(d_L + 1) + \log D_\tau + \log |P_\tau|_v + h_L \\ &\ll \log |P|_v + T(h_L + \log(D + T \log d_L)) + N \log(d_L + 1) + h_L \\ &\ll \log |P|_v + (T + 1)(h_L + \log(D + (T + 1)d_L)) \end{aligned}$$

for all $v \in M_K$, and the lemma follows. \square

Let k be a non-negative integer. We define $\mathcal{L}(k)$ as the sum of images of $\mathcal{O}_K[\xi_1, \dots, \xi_N]$ under all differentials of order $\leq k$, i.e.

$$\mathcal{L}(k) := \sum_{t \in \mathbb{Z}_{\geq 0}^n; |t| \leq k} L^t(\mathcal{O}_K[\xi_1, \dots, \xi_N]).$$

Let $\mathcal{I}(k)$ be the ideal $(\mathcal{O}_K[\xi_1, \dots, \xi_N] : \mathcal{L}(k))$ in \mathcal{O}_K . We get the following lemma.

Lemma 3.10.

$$\mathcal{I}(k) \supset (\mathcal{I}_L)^k, \quad \forall k \in \mathbb{N}.$$

Proof. We shall prove this by induction on k . If $k = 0$, the lemma is trivially true. Assume it is also true for $k = m \geq 0$. One has to show that

$$a_1 \cdots a_{m+1} L^t(\xi_i) \in \mathcal{O}_K[\xi_1, \dots, \xi_N]$$

for $i = 1, \dots, n$, for $a_1, \dots, a_{m+1} \in \mathcal{I}_L$ and for $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ with $|t| = m + 1$. There is at least one $j \in \{1, \dots, n\}$ such that $t_j \geq 1$. Put $\tau = (t_1, \dots, t_{j-1}, t_j - 1, t_{j+1}, \dots, t_n)$. We see that

$$a_1 \cdots a_{m+1} L^t(\xi_i) = a_1 \cdots a_m L^\tau(a_{m+1} L(j)(\xi_i)).$$

Since $a_{m+1} \in \mathcal{I}_L$ it follows that

$$a_{m+1} L(j)(\xi_i) = Q_{i,j}(\xi_1, \dots, \xi_N), \quad \forall i = 1, \dots, N$$

for some polynomials $Q_{i,j}(T_1, \dots, T_N)$ with coefficients in \mathcal{O}_K . By induction with $|\tau| = m$, we have $a_1 \cdots a_m \in \mathcal{I}_L^m \subset \mathcal{I}(m)$. In particular,

$$a_1 \cdots a_m L^\tau(Q_{i,j}(\xi_1, \dots, \xi_N)) \in \mathcal{O}_K[\xi_1, \dots, \xi_N].$$

The lemma is therefore proved. \square

Lemma 3.11. For $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ with $|t| = T$ and for a polynomial $P(T_1, \dots, T_N) \in \mathcal{O}_K[T_1, \dots, T_N]$ we have

$$\delta_L^T \partial^t P(f_1, \dots, f_N) \in \mathcal{O}_K[f_1, \dots, f_N].$$

Hence $\delta_L^T \partial^t f_i(0) \in \mathcal{O}_K$ for $i = 1, \dots, N$.

Proof. There exists a polynomial $P_t(T_1, \dots, T_N)$ with coefficients in K such that

$$L^t P(\xi_1, \dots, \xi_N) = P_t(\xi_1, \dots, \xi_N).$$

By Lemma 3.10, we see that the polynomial $\delta_L^T P_t$ has coefficients in \mathcal{O}_K . Note that

$$\partial^t P(f_1, \dots, f_N) = P_t(f_1, \dots, f_N),$$

and then one gets

$$\delta_L^T \partial^t P(f_1, \dots, f_N) \in \mathcal{O}_K[f_1, \dots, f_N].$$

Finally, since $f_i(0) = 0$ for $i = 1, \dots, N$ it follows that

$$\delta_L^T \partial^t f_i(0) = P_t(f_1(0), \dots, f_N(0)) \in \mathcal{O}_K, \quad \forall i = 1, \dots, N.$$

□

Proposition 3.12. The functions f_i satisfy

$$|f_i(x)|_p < 1, \quad \forall x \in B^n(|\delta_L|_p r_p).$$

Proof. It follows from the previous lemma and by considering the Taylor expansion of f_i at 0 together with the fact $|n!|_p \geq r_p^{n-1}$ for all positive integers n . □

3.6. The order of vanishing of analytic functions. In this section let F denote a complete subfield of \mathbb{C}_p . Let V be a vector subspace of $\text{Lie}(G(F))$ and f a non-zero p -adic analytic function on a neighborhood of a point $z \in F^n$. We say that f has a zero at z of order $\geq T$ along V if $(v_1 \cdots v_k f)(z) = 0$ for any $0 \leq k < T$ and for any $v_1, \dots, v_k \in V$. We also say that f has a zero at z of exact order T along V if it has order $\geq T$ at z along V and furthermore, there are w_1, \dots, w_T in V such that $(w_1 \cdots w_T f)(z) \neq 0$.

Proposition 3.13. With notations as above, let d be the dimension of V and let $\Delta_1, \dots, \Delta_d$ be a basis for V . Then f has a zero at z of order $\geq T$ along V if and only if $(\Delta_1^{t_1} \cdots \Delta_d^{t_d} f)(z) = 0$ for $(t_1, \dots, t_d) \in \mathbb{N}^d$ with $t_1 + \cdots + t_d < T$ and f has a zero at z of exact order T if it has order $\geq T$ at z along V and furthermore, there is a d -tuple $\tau = (\tau_1, \dots, \tau_d) \in \mathbb{N}^d$ such that $|\tau| = T$ and $(\Delta_1^{\tau_1} \cdots \Delta_d^{\tau_d} f)(z) \neq 0$.

Proof. We prove the first statement. In fact, it suffices to show that if $(\Delta_1^{t_1} \cdots \Delta_d^{t_d} f)(z) = 0$ for any $(t_1, \dots, t_d) \in \mathbb{N}^d$ with $t_1 + \cdots + t_d < T$, then f has a zero at z of order $\geq T$ along V . Let k be integer such that $0 \leq k < T$ and v_1, \dots, v_k arbitrary elements of V . For $i = 1, \dots, k$ one can write $v_i = a_{i1}\Delta_1 + \cdots + a_{id}\Delta_d$ with $a_{i1}, \dots, a_{id} \in F$. For $t = (t_1, \dots, t_d) \in \mathbb{N}^d$ with $|t| < T$, we expand

$$(v_1 \cdots v_k f)(z) = \left(\prod_{i=1}^k (a_{i1}\Delta_1 + \cdots + a_{id}\Delta_d) f \right)(z) = \sum_{\alpha \in I} a_\alpha (\Delta_1^{\alpha_1} \cdots \Delta_d^{\alpha_d} f)(z).$$

Since $k < T$ it follows that $|\alpha| = \alpha_1 + \cdots + \alpha_d < T$ for every $\alpha \in I$. Hence the sum vanishes, and this shows the first statement. It is clear that the second statement follows at once from the definition and the first statement. \square

4. PROOFS

4.1. Proof of the second statement of Theorem 2.1. We shall show that the first assertion of the theorem implies the second one. Let $u \in \Lambda_v$ such that $\text{Exp}(u)$ is an algebraic point in $G(K)$. We define

$$n(u) := \max \left\{ 0, \left\lfloor \frac{1}{p-1} - v(u) \right\rfloor + 1 \right\},$$

and $u' := p^{n(u)}u$. Then $u' \in \Lambda_v$ and

$$|u'|_p = |p^{n(u)}u|_p = p^{-n(u)-v(u)} = p^{\frac{1}{p-1}-v(u)-n(u)} r_p < r_p.$$

Moreover, if $l(u) \neq 0$ then $l(u') = p^{n(u)}l(u) \neq 0$ and applying the first statement of Theorem 2.1 to u' in $\Lambda_v \cap B^n(r_p|\delta_L|_p)$ one gets

$$\log |l(u')|_p > -c_0 \omega_L^{n+3} b h'^n (\log b + \log h')^{n+3} \log p;$$

here $h' := \max\{1, h(\gamma')\}$ with $\gamma' := \text{Exp}(u') = p^{n(u)}\text{Exp}(u) = \gamma^{p^{n(u)}}$ where $\gamma := \text{Exp}(u)$. By [19, Prop. 5] one has

$$h(\gamma^{p^{n(u)}}) \leq (p^{n(u)})^2 h(\gamma) \leq p^{2n(u)} h$$

and this implies that $h' \leq p^{2n(u)} h$. Hence

$$n(u) \log p + \log |l(u)|_p > -c_0 \omega_L^{n+3} b h^n (\log b + \log h + 2n(u) \log p)^{n+3} \log p.$$

We therefore conclude that

$$\log |l(u)|_p > -c_1 \omega_L^{n+3} b h^n (\log b + \log h + 2n(u) \log p)^{n+3} \log p$$

for some positive constant c_1 .

4.2. A projective embedding. Following [19] (cf. also [9] and [24]), there exist a positive integer N and an embedding $\varphi : G \hookrightarrow \mathbb{P}^N$ of the group from above, which is defined over a number field K of degree m . Without loss of generality, we may assume that the identity element $e \in G(K)$ under φ has coordinates $(1 : 0 : \dots : 0)$ in \mathbb{P}^N .

Lemma 4.1. There exists an embedding $\psi : G \rightarrow \mathbb{P}^N$ defined over a number field of degree $m(N+1)$ such that $\psi(e) = (1 : 0 : \dots : 0)$ and $X_0(\psi(g)) \neq 0$ for all $g \in G(K)$, where X_0 denotes the first projective coordinate on \mathbb{P}^N .

Proof. We choose a field extension K_1 of K of degree $N+1$ and a basis $\epsilon_0, \dots, \epsilon_N$ of K_1 over K . The degree of the extension $K_1 \supseteq \mathbb{Q}$ is therefore $m(N+1)$. It is clear that the vectors

$$(\epsilon_0, 0, \dots, 0), (-\epsilon_1, \epsilon_0, 0, \dots, 0), \dots, (-\epsilon_N, 0, \dots, 0, \epsilon_0)$$

form a basis of K_1^{N+1} which gives rise to a unique element in $\mathrm{GL}_{N+1}(K_1)$ mapping this basis to the standard basis of K_1^{N+1} . This linear isomorphism is expressed explicitly by the matrix

$$A = \begin{pmatrix} \epsilon_0^{-1} & \epsilon_0^{-2}\epsilon_1 & \dots & \epsilon_0^{-2}\epsilon_N \\ 0 & \epsilon_0^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \epsilon_0^{-1} \end{pmatrix}.$$

We let ψ be the composition of A with the embedding φ above. Then $\psi(e)$ has projective coordinates $(1 : 0 : \dots : 0)$ and $X_0(\psi(g)) \neq 0$ for all $g \in G(K)$. Indeed, let $(x_0 : x_1 : \dots : x_N)$ be a projective coordinate of $\varphi(g)$. By the construction of ψ , we obtain

$$\begin{aligned} \psi(g) &= (\epsilon_0^{-1}x_0 + \epsilon_0^{-2}\epsilon_1x_1 + \dots + \epsilon_0^{-2}\epsilon_Nx_N : \epsilon_0^{-1}x_1 : \dots : \epsilon_0^{-1}x_N) \\ &= (\epsilon_0x_0 + \epsilon_1x_1 + \dots + \epsilon_Nx_N : \epsilon_0x_1 : \dots : \epsilon_0x_N). \end{aligned}$$

Thus we see that $\psi(e) = (1 : 0 : \dots : 0)$. In addition, since $\epsilon_0, \dots, \epsilon_N$ is a basis of K_1 over K and x_0, \dots, x_N are in K , not all zero, it follows that $X_0(\psi(g))$ is non-zero. Note that the embedding ψ is defined over K_1 . \square

We shall fix the embedding $\psi : G \hookrightarrow \mathbb{P}^N$ for the rest of the paper and identify each element $g \in G$ with its image $\psi(g)$ in \mathbb{P}^N . By [23, Section 2], there is a finite field extension K_2 of K_1 (the degree of this extension is a positive constant) with the following property: There exist bihomogeneous polynomials E_0, \dots, E_N in Z_0, \dots, Z_N and X_0, \dots, X_N of bidegree (b, b) with coefficients in K_2 and their height bounded from above by a positive constant, and a Zariski open set $U \subset G \times G$ containing $\Gamma(\gamma) \times \Gamma(\gamma)$ such that for $(g, g') \in U$ the homogeneous coordinates of $g + g'$

are $(E_0(g, g') : \dots : E_N(g, g'))$; here $\Gamma(\gamma)$ denotes the subgroup generated by γ in $G(K)$ with $\gamma := \text{Exp}(u)$. The degree of the extension K_2 over K is also a positive constant. We may therefore assume, without loss of generality, that K is already equal to K_2 and has degree d over \mathbb{Q} . We call (E_1, \dots, E_N) an addition formula for G and from now on we fix such an addition formula $E = (E_1, \dots, E_N)$.

4.3. Basis of the hyperplane. We define the linear form in $n+1$ variables

$$\mathcal{L}(Z_0, Z_1, \dots, Z_n) := Z_0 - l(Z_1, \dots, Z_n).$$

This gives the vector space

$$\mathcal{W} := \{(z_0, z_1, \dots, z_n) \in K_v^{n+1}; z_0 = l(z_1, \dots, z_n)\} \subset K_v^{n+1}.$$

Let e_1, \dots, e_n be the basis for \mathcal{W} defined by

$$e_1 = (\beta_1, 1, 0, \dots, 0), e_2 = (\beta_2, 0, 1, 0, \dots, 0), \dots, e_n = (\beta_n, 0, \dots, 0, 1).$$

This gives differential operators (corresponding to the isomorphism ∂ introduced in Section 3.5)

$$\Delta_1 = \partial(e_1) = \beta_1 \partial_0 + \partial_1, \Delta_2 = \partial(e_2) = \beta_2 \partial_0 + \partial_2, \dots, \Delta_n = \partial(e_n) = \beta_n \partial_0 + \partial_n;$$

here $\partial_0, \dots, \partial_n$ is the standard basis for $\text{Lie}(K_v^{n+1})$. Let $\mathbf{u}_0 := (0, u_1, \dots, u_n)$ and $\mathbf{u} := (u_0, u_1, \dots, u_n)$ be vectors in K_v^{n+1} with $u_0 := l(u)$. Then

$$\mathbf{u} = u_1 e_1 + \dots + u_n e_n$$

and this shows that $\mathbf{u} \in \mathcal{W}$. We furthermore see that

$$\mathbf{u} - \mathbf{u}_0 = (l(u), 0, \dots, 0).$$

Define

$$\Delta^t := \Delta_1^{t_1} \dots \Delta_n^{t_n}$$

for $t = (t_1, \dots, t_n) \in \mathbb{N}^n$.

4.4. The auxiliary function. In this section we shall construct an auxiliary polynomial by using Siegel's lemma. Let $\mathcal{G} := \mathbb{G}_a \times G$ be the product of the additive group \mathbb{G}_a with G . The exponential map of the Lie group $\mathcal{G}(K_v)$ is $\exp_{\mathcal{G}(K_v)} = \text{id}_{K_v} \times \exp$. Note that for $u \in \Lambda_v$ we have $X_0(\text{Exp}(u)) \neq 0$; here the map $\text{Exp} : \Lambda_v \rightarrow G(K_v)$ is defined in Section 3.5. We introduce the function

$$\Psi_P := (\text{id}_{K_v} \times \text{Exp})^* P\left(Y, 1, \frac{X_1}{X_0}, \dots, \frac{X_N}{X_0}\right)$$

for each polynomial P in $N+2$ variables Y, X_0, \dots, X_N . This means that $\Psi_P(w) = P(y, 1, f_1(x_1, \dots, x_n), \dots, f_N(x_1, \dots, x_n))$ is analytic on $K_v \times \Lambda_v^n$, where $w = (y, x) \in K_v^{n+1}$ with $x = (x_1, \dots, x_n) \in \Lambda_v^n$.

We define the order $\text{ord}_{g, \mathcal{W}} P$ of P at $g = (\text{id}_{K_v} \times \text{Exp})(w)$ along \mathcal{W} to be infinity if Ψ_P is identically zero in a neighborhood of x , and to be the order of Ψ_P at w along \mathcal{W} , otherwise.

Let S_0, D_0, D, T be positive integers. We apply Siegel's lemma to construct a polynomial P in $N + 2$ variables with coefficients in \mathcal{O}_K such that P does not vanish identically on \mathcal{G} and has height $h(P)$ bounded from above by a quantity in terms of L, S_0, D_0, D, T, b, h . We further require that $\text{ord}_{s\mathbf{u}_0, \mathcal{W}} \Psi_P \gg T$ for all $0 \leq s < S_0$.

Proposition 4.2. There are positive constants c_2 and c_3 such that if $D_0 D^n \geq c_2 S_0 T^n$ there is a polynomial P in $N + 2$ variables Y, X_0, \dots, X_N with coefficients in \mathcal{O}_K , homogeneous in X_0, \dots, X_N of degree D , and with $\deg P_Y \leq D_0$ such that

1. P does not vanish identically on \mathcal{G} ,
2. $(\Delta^t \Psi_P)(s\mathbf{u}_0) = 0, 0 \leq s < S_0, t = (t_1, \dots, t_n), 0 \leq t_1, \dots, t_n < 2T$,
3. $h(P) \leq c_3(T(h_L + \log \delta_L + \log(D + T \log d_L)) + D_0 b + D S_0^2 h)$.

Proof. Since the dimension of G is n , without loss of generality, we may assume that the homogeneous coordinates X_0, \dots, X_n are algebraically independent modulo the ideal of G . We shall construct a non-zero polynomial P in $n + 2$ variables Y and X_0, \dots, X_n which is homogeneous in X_0, \dots, X_n of degree D (this polynomial therefore satisfies 1. in the proposition) such that $\deg_Y P \leq D_0$ and such that 2. and 3. in the proposition are satisfied. Such a polynomial can be written in the form

$$P(Y, X) = \sum_{i=0}^{D_0} \sum_{j=1}^{D_1} p_{ij} Y^i M_j(X_0, \dots, X_n),$$

where D_1 is the number of homogeneous monomials of degree D in the $n + 1$ variables X_0, \dots, X_n and M_1, \dots, M_{D_1} run through all these monomials. An easy computation shows that $D_1 = \binom{D+n}{n}$. For short, we write Ψ for Ψ_P . Let $E = (E_1, \dots, E_N)$ be the addition formula for G from above. By abuse of notation, we put

$$E_i(z, x) := E_i(1, f_1(z), \dots, f_N(z), 1, f_1(x), \dots, f_N(x)),$$

for z, x in Λ_v . For $y \in K_v$ we also define

$$\Psi_s(y, x) := \Psi(y, su + x) E_0(su, x)^D.$$

Put

$$I := \{(s, t); 0 \leq s < S_0, t = (t_1, \dots, t_n), 0 \leq t_1, \dots, t_n < 2T\}.$$

For any $(s, t) \in I$ we shall determine the coefficients p_{ij} such that

$$(\Delta^t \Psi_s)(0, 0) = 0, \quad \forall (s, t) \in I.$$

By the property of the addition formula E , for any x in a neighbourhood of 0 small enough so that $E(su, x) \neq 0$, one gets

$$f_i(su + x) = \frac{E_i(su, x)}{E_0(su, x)}, \quad i = 1, \dots, N.$$

This leads to

$$\begin{aligned} M_j(1, f_1(su + x), \dots, f_n(su + x)) \\ &= M_j\left(1, \frac{E_1(su, x)}{E_0(su, x)}, \dots, \frac{E_n(su, x)}{E_0(su, x)}\right) \\ &= E_0(su, x)^{-D} M_j(E_0(su, x), \dots, E_n(su, x)). \end{aligned}$$

Therefore one gets

$$\Psi_s(y, x) = \Psi(y, su + x) E_0(su, x)^D = \sum_{i,j} p_{ij} y^i M_j(E_0(su, x), \dots, E_n(su, x)).$$

On the other hand, for each s , we can express $E_i(su, x)$ as

$$E_i(su, x) = F_i(f_1(x), \dots, f_N(x)), \quad i = 0, \dots, n,$$

here F_i are polynomials in N variables with polynomials (which have coefficients in K) in the $f_1(su), \dots, f_N(su)$ as coefficients. Since

$$\gamma^s = \text{Exp}(su) = (1 : f_1(su) : \dots : f_N(su))$$

and since $h(\gamma^s) \ll s^2 h$ (see [19, Prop. 5]) we may estimate the height of polynomials $h(F_i) \ll s^2 h$ for $i = 0, \dots, n$. One can therefore choose a common denominator $d_s \ll s^2 h$ for the polynomials F_0, \dots, F_n . Since M_j is a monomial of degree D , there is a polynomial $Q_{j,s}$ in N variables of degree $\ll D$ with $\log |Q_{j,s}|_v \ll D s^2 h$ for $v \in M_K$ such that

$$M_j(E_0(su, x), \dots, E_n(su, x)) = Q_{j,s}(f_1(x), \dots, f_N(x))$$

for each $j = 1, \dots, D_1$. Then

$$\Psi_s(y, x) = \sum_{i,j} p_{ij} y^i Q_{j,s}(f_1(x), \dots, f_N(x)),$$

this gives

$$(\Delta^t \Psi_s)(0, 0) = \sum_{i,j} p_{ij} (\Delta^t (y^i Q_{j,s}(f_1, \dots, f_N)))(0, 0).$$

Define

$$a_{ij}^{st} := (\Delta^t (y^i Q_{j,s}(f_1, \dots, f_N)))(0, 0)$$

for $i = 0, \dots, D_0$, $j = 1, \dots, D_1$ and $(s, t) \in I$. Note that $\partial_0 = \partial/\partial y$. We expand

$$\begin{aligned}
 a_{i,j}^{st} &= (\Delta_1^{t_1} \cdots \Delta_n^{t_n} (y^i Q_{j,s}(f_1, \dots, f_N))) (0, 0) \\
 &= ((\beta_1 \partial_0 + \partial_1)^{t_1} \cdots (\beta_n \partial_0 + \partial_n)^{t_n} (y^i Q_{j,s}(f_1, \dots, f_N))) (0, 0) \\
 &= \sum_{i_1=0}^{t_1} \cdots \sum_{i_n=0}^{t_n} \binom{t_1}{i_1} \cdots \binom{t_n}{i_n} \beta_1^{t_1-i_1} \cdots \beta_n^{t_n-i_n} \\
 &\quad \cdot \left(\left(\frac{\partial}{\partial y} \right)^{(t_1+\cdots+t_n)-(i_1+\cdots+i_n)} \partial_1^{i_1} \cdots \partial_n^{i_n} (y^i Q_{j,s}(f_1, \dots, f_N)) \right) (0, 0) \\
 &= \sum_{i_1=0}^{t_1} \cdots \sum_{i_n=0}^{t_n} \binom{t_1}{i_1} \cdots \binom{t_n}{i_n} \beta_1^{t_1-i_1} \cdots \beta_n^{t_n-i_n} \\
 &\quad \cdot \left(\left(\frac{\partial}{\partial y} \right)^{(t_1+\cdots+t_n)-(i_1+\cdots+i_n)} y^i \right) (0) \left(\partial_1^{i_1} \cdots \partial_n^{i_n} (Q_{j,s}(f_1, \dots, f_N)) \right) (0).
 \end{aligned}$$

For $m \in \mathbb{N}^n$ we obtain from Lemma 3.9

$$\partial^m (Q_{j,s}(f_1, \dots, f_N)) = Q_{j,s,m}(f_1, \dots, f_N)$$

for some polynomial $Q_{j,s,m}$ in N variables with

$$\begin{aligned}
 \log |Q_{j,s,m}|_v &\ll \log |Q_{j,s}|_v + |m|(h_L + \log(D + |m|d_L)) \\
 &\ll |m|(h_L + \log(D + |m|d_L)) + Ds^2h, \quad \forall v \in M_K.
 \end{aligned}$$

This means that

$$\log \left| \left(\partial^m (Q_{j,s}(f_1, \dots, f_N)) \right) (0) \right|_v \ll |m|(h_L + \log(D + |m|d_L)) + Ds^2h$$

for $v \in M_K$. In particular,

$$\begin{aligned}
 \log \left| \left(\partial_1^{i_1} \cdots \partial_n^{i_n} (Q_{j,s}(f_1, \dots, f_N)) \right) (0) \right|_v \\
 \ll T(h_L + \log(D + Td_L)) + Ds^2h, \quad \forall v \in M_K.
 \end{aligned}$$

Furthermore one gets

$$\left(\left(\frac{\partial}{\partial y} \right)^{(t_1+\cdots+t_n)-(i_1+\cdots+i_n)} y^i \right) (0) = \begin{cases} 0 & \text{if } (t_1 + \cdots + t_n) - (i_1 + \cdots + i_n) \neq i, \\ i! & \text{if } (t_1 + \cdots + t_n) - (i_1 + \cdots + i_n) = i. \end{cases}$$

In other words, we have

$$\log \left| \left(\left(\frac{\partial}{\partial y} \right)^{(t_1+\cdots+t_n)-(i_1+\cdots+i_n)} y^i \right) (0) \right|_v \ll \log(T!) \ll T \log T, \quad \forall v \in M_K^\infty.$$

We deduce that

$$\log |a_{i,j}^{st}|_v \ll T(h_L + \log(D + Td_L)) + Ds^2h, \quad \forall v \in M_K^\infty.$$

Since $h(\beta_i) \leq b$ for $i = 1, \dots, n$, $\log |\beta_i|_v \leq b$ for $v \in M_K$. By noting that $d_s \delta_L^{|m|} Q_{j,s,m}$ has coefficients in \mathcal{O}_K , we get $d_s \delta_L^{2nT} a_{i,j}^{st}$ is also in \mathcal{O}_K and the

quantity $\log |d_s \delta_L^{2nT} a_{ij}^{st}|_v$ is

$$\ll D_0 b + T(\log \delta_L + h_L + \log(D + T \log d_L)) + DS_0^2 h$$

for $(s, t) \in I$ and for $v \in M_K^\infty$. We now consider the linear forms in $n_0 := D_0 D_1$ variables T_{ij}

$$l_{st} := \sum_{i,j} b_{ij}^{st} T_{ij},$$

where $b_{ij}^{st} := d_s \delta_L^{2nT} a_{ij}^{st}$ for all $(s, t) \in I$. Let m_0 be the number of these linear forms, then $m_0 \ll S_0 T^n$ and $n_0 = D_0 D_1 = D_0 \binom{D+n}{n} \gg D_0 D^n$. Since $b_{ij}^{st} \in \mathcal{O}_K$ we get

$$\begin{aligned} h_{\max}(l_{st}) &= \sum_{v \in M_K^\infty} \log \max_{i,j} |b_{ij}^{st}|_v \\ &\ll D_0 b + T(h_L + \log \delta_L + \log(D + T \log d_L)) + DS_0^2 h. \end{aligned}$$

We now apply Siegel's lemma. It follows that under the condition $D_0 D^n \gg S_0 T^n$ there is a non-zero vector $p_0 = (p_{ij})$ with coordinates in \mathcal{O}_K such that $l_{st}(p_0) = 0$ and

$$h(p_0) \leq \frac{m_0}{n_0 - m_0} \max_{s,t} h_{L^2}(l_{st})$$

But using

$$h_{L^2}(l_{st}) \ll h_{\max}(l_{st}) + \log n_0,$$

this gives

$$h(P) \ll D_0 b + T(h_L + \log \delta_L + \log(D + T d_L)) + DS_0^2 h.$$

It remains to show that $(\Delta^t \Psi)(s\mathbf{u}_0) = 0$. In fact, since $l_{st}(p_0) = 0$ one gets $(\Delta^t \Psi_s)(0, 0) = 0$ for $(s, t) \in I$. Put

$$\Psi_s^*(y, x) := \Psi(y, su + x), \quad E_s(x) := E_0(su, x)^D,$$

then $\Psi_s^* = \Psi_s E_s^{-D}$. We therefore get by Leibnitz' rule for derivation that

$$(\Delta^t \Psi)(s\mathbf{u}_0) = (\Delta^t \Psi)(0, su) = (\Delta^t \Psi_s^*)(0, 0) = (\Delta^t (\Psi_s E_s^{-D}))(0, 0) = 0.$$

This completes the proof. \square

From now on until Section 4.8, we shall fix a polynomial P as in Proposition 4.2 and let $\Psi = \Psi_P$ be the analytic function associated with P .

4.5. Extrapolation. In this section we use the p -adic Schwarz lemma to give an upper bound for $|(\Delta^t \Psi)(s\mathbf{u})|_p$ (with $|t| < T$). We need the following lemma.

Lemma 4.3. Let Q be a polynomial in $k + 1$ variables X_0, \dots, X_k with coefficients in the ring \mathcal{O}_v of algebraic integers of K_v and $\deg_{X_0} Q \leq l$ with $l \in \mathbb{N}, l \geq 1$. Then

$$|Q(x_0, x) - Q(0, x)|_p \leq \max_{1 \leq i \leq l} |x_0|_p^i,$$

for any $x_0 \in K_v$ and $x \in \mathcal{O}_v^k$.

Proof. We define the polynomial $Q_x(X) := Q(X, x)$ in one variable X . By assumption and by the ultrametric inequality, we see that Q_x has coefficients in \mathcal{O}_v . We write $Q_x(X) = a_l X^l + \dots + a_0$, with $a_0, \dots, a_l \in \mathcal{O}_v$. Then

$$|Q_x(x_0) - Q_x(0)|_p = |a_l x_0^l + \dots + a_1 x_0|_p \leq \max_{1 \leq i \leq l} |a_i x_0^i|_p \leq \max_{1 \leq i \leq l} |x_0|_p^i,$$

and the lemma follows. \square

Lemma 4.4. For $0 \leq s < S$ and for $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ such that $0 \leq t_1, \dots, t_n < 2T$ we have

$$|(\Delta^t \Psi)(s\mathbf{u}) - (\Delta^t \Psi)(s\mathbf{u}_0)|_p \leq |\delta_L^{-1}|_p^{2nT} |l(u)|_p.$$

Proof. In fact, we can write again

$$P(Y, X_0, \dots, X_N) = \sum_{i,j} p_{ij} Y^i M_j(X_0, \dots, X_N).$$

Put $R_j(x) = M_j(1, f_1(x), \dots, f_N(x))$, then

$$\begin{aligned} \Delta^t \Psi &= \sum_{i,j} p_{ij} (\Delta^t (y^i R_j)) \\ &= \sum_{i,j} p_{ij} ((\beta_1 \partial_0 + \partial_1)^{t_1} \dots (\beta_n \partial_0 + \partial_n)^{t_n} (y^i R_j)) \\ &= \sum_{i,j} p_{ij} \sum_{i_1=0}^{t_1} \dots \sum_{i_n=0}^{t_n} \binom{t_1}{i_1} \dots \binom{t_n}{i_n} \beta_1^{t_1-i_1} \dots \beta_n^{t_n-i_n} \\ &\quad \cdot \left(\left(\frac{\partial}{\partial y} \right)^{(t_1+\dots+t_n)-(i_1+\dots+i_n)} \partial_1^{i_1} \dots \partial_n^{i_n} (y^i R_j) \right) \\ &= \sum_{i,j} p_{ij} \sum_{i_1=0}^{t_1} \dots \sum_{i_n=0}^{t_n} \binom{t_1}{i_1} \dots \binom{t_n}{i_n} \beta_1^{t_1-i_1} \dots \beta_n^{t_n-i_n} \\ &\quad \cdot \left(\left(\frac{\partial}{\partial y} \right)^{(t_1+\dots+t_n)-(i_1+\dots+i_n)} y^i (\partial_1^{i_1} \dots \partial_n^{i_n} R_j) \right). \end{aligned}$$

Using the fact that

$$\left(\frac{\partial}{\partial y} \right)^{(t_1+\dots+t_n)-(i_1+\dots+i_n)} y^i = 0 \quad \text{if} \quad (t_1 + \dots + t_n) - (i_1 + \dots + i_n) > i,$$

and that $\partial_1^{i_1} \cdots \partial_n^{i_n} R_j$ is a polynomial in f_1, \dots, f_N with its denominator bounded from above by $\delta_L^{|t|}$ by Lemma 3.11, we deduce that

$$\delta_L^{|t|} \beta_1^{t_1-i_1} \cdots \beta_n^{t_n-i_n} \left(\left(\frac{\partial}{\partial y} \right)^{(t_1+\cdots+t_n)-(i_1+\cdots+i_n)} y^i \right) (\partial_1^{i_1} \cdots \partial_n^{i_n} R_j)$$

is a polynomial in y, f_1, \dots, f_N with coefficients in \mathcal{O}_K . On the other hand, the coefficients p_{ij} are in \mathcal{O}_K , and this implies that

$$\delta_L^{|t|} (\Delta^t \Psi) = Q_t(y, f_1, \dots, f_N),$$

for some polynomial $Q_t(Y, X_1, \dots, X_N)$ with coefficients in \mathcal{O}_K , and with $\deg_Y Q_t \leq D_0$. This means that $|(\Delta^t \Psi)(s\mathbf{u}) - (\Delta^t \Psi)(s\mathbf{u}_0)|_p$ is equal to

$$|\delta_L^{-|t|} Q_t(su_0, f_1(su), \dots, f_N(su)) - Q_t(0, f_1(su), \dots, f_N(su))|_p.$$

Since $u \in \Lambda_v \cap B^n(r_p |\delta_L|_p)$, we get $|f_1(su)|_p, \dots, |f_N(su)|_p < 1$ by Proposition 3.12 and, taking into account that $r_p |\delta_L|_p < 1$ and that $\beta_1, \dots, \beta_n \in \mathcal{O}_K$, we find that

$$|su_0|_p = |s|_p |u_0|_p \leq |u_0|_p = |\beta_1 u_1 + \cdots + \beta_n u_n|_p < 1.$$

By Lemma 4.3 we obtain

$$\begin{aligned} |(\Delta^t \Psi)(s\mathbf{u}) - (\Delta^t \Psi)(s\mathbf{u}_0)|_p &\leq |\delta_L|_p^{-2nT} \max_{1 \leq i \leq D_0} |su_0|_p^i \\ &\leq |\delta_L|_p^{-1} |u_0|_p^{2nT} = |\delta_L|_p^{-1} |l(u)|_p^{2nT}. \end{aligned}$$

This gives the assertion. \square

Proposition 4.5. For $0 \leq s < S_0$ and for $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ such that $0 \leq t_1, \dots, t_n < 2T$ the estimate

$$|(\Delta^t \Psi)(s\mathbf{u})|_p \leq |\delta_L|_p^{-1} |l(u)|_p^{2nT}$$

holds.

Proof. By Proposition 4.4, for $0 \leq s < S_0$ and for $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ with $0 \leq t_1, \dots, t_n < 2T$ one has

$$|(\Delta^t \Psi)(s\mathbf{u}) - (\Delta^t \Psi)(s\mathbf{u}_0)|_p \leq |\delta_L|_p^{-1} |l(u)|_p^{2nT}.$$

Moreover by Proposition 4.2, for $0 \leq s < S_0$ and for $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ with $0 \leq t_1, \dots, t_n < 2T$, we have

$$(\Delta^t \Psi)(s\mathbf{u}_0) = 0.$$

This gives

$$|(\Delta^t \Psi)(s\mathbf{u})|_p \leq |\delta_L|_p^{-1} |l(u)|_p^{2nT}$$

as claimed. \square

For each n -tuple $t \in \mathbb{N}^n$ such that $|t| < T$ we introduce the function

$$f(z) := (\Delta^t \Psi)(z\mathbf{u})$$

in the variable z . It is analytic on $\overline{B}(1)$. Our next step is to apply Proposition 3.3 to the function f . We shall prove an upper bound for the derivatives of f on a certain finite set. Thanks to Proposition 4.5, one gets

Proposition 4.6. For $\tau, s \in \mathbb{Z}$ such that $0 \leq \tau < T$ and $0 \leq s < S_0$ one has

$$|f^{(\tau)}(s)|_p \leq |\delta_L^{-1}|_p^{2nT} |l(u)|_p.$$

Proof. By recalling that $\mathbf{u} = u_1 e_1 + \cdots + u_n e_n$ and using the composition rule for derivatives we get

$$\begin{aligned} f^{(\tau)}(z) &= ((u_0 \partial_0 + \cdots + u_n \partial_n)^\tau \Delta^t \Psi)(z\mathbf{u}) \\ &= \left(((\beta_1 u_1 + \cdots + \beta_n u_n) \partial_0 + u_1 \partial_1 + \cdots + u_n \partial_n)^\tau \Delta^t \Psi \right)(z\mathbf{u}) \\ &= \left((u_1(\beta_1 \partial_0 + \partial_1) + \cdots + u_n(\beta_n \partial_0 + \partial_n))^\tau \Delta^t \Psi \right)(z\mathbf{u}) \\ &= ((u_1 \Delta_1 + \cdots + u_n \Delta_n)^\tau \Delta^t \Psi)(z\mathbf{u}). \end{aligned}$$

Since $|u_i|_p < 1$ for $i = 1, \dots, n$, the multinomial expansion together with the ultrametric inequality gives

$$|f^{(\tau)}(z)|_p \leq \max_{0 \leq i_1, \dots, i_n \leq \tau; i_1 + \dots + i_n = \tau} |(\Delta_1^{i_1} \cdots \Delta_n^{i_n} \Delta^t \Psi)(z\mathbf{u})|_p.$$

Since τ and $|t|$ are $< T$ the assertion follows from Proposition 4.5. \square

Lemma 4.7. Let α be a non-zero element in K_v such that $|\alpha|_p < p^{-1/(p-1)}$. Then

$$v(\alpha) - \frac{1}{p-1} \geq \frac{1}{2d^2}.$$

Proof. We know that $v(K_v^\times) = \frac{1}{d_v} \mathbb{Z}$, with $d_v := [K_v : \mathbb{Q}_p]$. Since $|\alpha|_p = p^{-v(\alpha)} < p^{-\frac{1}{p-1}}$ there is a positive integer a such that

$$v(\alpha) = \frac{a}{d_v} > \frac{1}{p-1}.$$

This implies that $a(p-1) - d_v \geq 1$. If $p-1 \geq 2d_v$ then

$$v(\alpha) - \frac{1}{p-1} \geq \frac{1}{d_v} - \frac{1}{p-1} \geq \frac{1}{2d_v} \geq \frac{1}{2d} \geq \frac{1}{2d^2}.$$

Otherwise, if $p-1 < 2d_v$ then

$$v(\alpha) - \frac{1}{p-1} = \frac{a(p-1) - d_v}{d_v(p-1)} \geq \frac{1}{d_v(p-1)} > \frac{1}{2d_v^2} \geq \frac{1}{2d^2}.$$

\square

From now on we put $\epsilon := \frac{1}{3d^2}$. Combining Lemma 4.7 and Proposition 4.6 together with Proposition 3.3, we get the following result.

Proposition 4.8. For $s \in \mathbb{N}$ and for $t = (t_1, \dots, t_n) \in \mathbb{N}^n$ such that $|t| < T$ the quantity $|(\Delta^t \Psi)(s\mathbf{u})|_p$ is bounded from above by

$$p^{-(\epsilon S_0 - e_L)T} \max \left\{ 1, p^{((2n-1)e_L + \epsilon S_0 + \frac{1}{p-1})T} S_0^{S_0 T} |l(u)|_p \right\}.$$

Proof. As above, we consider the function $f(z) = (\Delta^t \Psi)(z\mathbf{u})$ in the variable z , and apply the p -adic Schwarz lemma to the function f . We first show that the function f is analytic on $\overline{B}(R)$, where $R := p^\epsilon$. It suffices to show that $zu_i \in B(r_p |\delta_L|_p)$ for $z \in \overline{B}(R)$ and for $i = 1, \dots, n$. In fact, if $u_i = 0$ then it is trivially true. Otherwise, since $|\delta_L^{-1} u_i|_p < p^{-\frac{1}{p-1}}$ it follows from Lemma 4.7 that

$$v(\delta_L^{-1} u_i) - \frac{1}{p-1} \geq \frac{1}{2d^2}.$$

Hence

$$v(\delta_L^{-1} u_i) - \epsilon = \frac{1}{p-1} + \left(v(\delta_L^{-1} u_i) - \frac{1}{p-1} - \frac{1}{3d^2} \right) > \frac{1}{p-1}$$

which leads to

$$R |\delta_L^{-1} u_i|_p = p^\epsilon p^{-v(\delta_L^{-1} u_i)} = p^{-(v(\delta_L^{-1} u_i) - \epsilon)} < p^{-\frac{1}{p-1}}$$

or equivalently to $R |u_i|_p < r_p |\delta_L|_p$. This means that $zu_i \in B(r_p |\delta_L|_p)$ for $z \in \overline{B}(R)$. Next we establish an upper bound for $|f|_R$. As in the proof of Proposition 4.4 there is a polynomial $Q(Y, X_1, \dots, X_N)$ with coefficients in \mathcal{O}_K such that $\deg_Y Q \leq D_0$ and

$$f(z) = \delta_L^{-T} Q(zu_0, f_1(zu), \dots, f_N(zu)).$$

We note that

$$|zu_0|_p = |\beta_1 zu_1 + \dots + \beta_n zu_n|_p \leq |zu_1 + \dots + zu_n|_p \leq \max\{|zu_1|_p, \dots, |zu_n|_p\} < 1$$

and deduce from Proposition 3.12 that $|f_i(zu)|_p < 1$ for $i = 1, \dots, N$ and for $z \in \overline{B}(R)$. This gives $|Q(zu_0, f_1(zu), \dots, f_N(zu))|_p \leq 1$ which leads to

$$|f(z)|_p \leq |\delta_L^{-1}|_p^T, \quad \forall z \in \overline{B}(R).$$

In other words we have

$$|f|_R \leq |\delta_L^{-1}|_p^T.$$

Finally let Γ be the set $\{s \in \mathbb{Z}; 0 \leq s < S_0\}$ and δ be the minimum of $|s - s'|_p$ for $s \neq s'$ in Γ . The cardinality of Γ is S_0 and we have $\delta \leq 1$. We define

$$\mu := \sup\{|f^{(\tau)}(s)|_p; 0 \leq \tau < T, s \in \Gamma\}.$$

Using Lemma 4.6 one gets $\mu \leq |\delta_L^{-1}|_p^{2nT} |l(u)|_p$. We apply Proposition 3.3 to the function f to obtain

$$\begin{aligned} |f|_1 &\leq \max \left\{ \left(\frac{1}{R} \right)^{S_0 T} |f|_R, \mu \left(\frac{1}{\delta} \right)^{S_0 T-1} r_p^{-(T-1)} \right\} \\ &\leq \max \left\{ p^{-\epsilon S_0 T} |\delta_L^{-1}|_p^T, |\delta_L^{-1}|_p^{2nT} |l(u)|_p \delta^{-(S_0 T-1)} r_p^{-T} \right\} \\ &\leq \max \left\{ p^{-\epsilon S_0 T} p^{e_L T}, p^{2ne_L T} p^{\frac{T}{p-1}} \delta^{-(S_0 T-1)} |l(u)|_p \right\} \\ &\leq \max \left\{ p^{-(\epsilon S_0 - e_L)T}, p^{(2ne_L + \frac{1}{p-1})T} \delta^{-(S_0 T-1)} |l(u)|_p \right\}. \end{aligned}$$

Moreover, for $s, s' \in \Gamma$ such that $s \neq s'$ one has

$$|s - s'|_p \geq \frac{1}{|s - s'|} > \frac{1}{S_0}.$$

This gives $\delta^{-1} < S_0$. Thus we obtain

$$\begin{aligned} |f|_1 &\leq \max \left\{ p^{-(\epsilon S_0 - e_L)T}, p^{(2ne_L + \frac{1}{p-1})T} S_0^{S_0 T} |l(u)|_p \right\} \\ &= p^{-(\epsilon S_0 - e_L)T} \max \left\{ 1, p^{((2n-1)e_L + \epsilon S_0 + \frac{1}{p-1})T} S_0^{S_0 T} |l(u)|_p \right\}. \end{aligned}$$

The proposition therefore follows from the fact that

$$|(\Delta^t \Psi)(s\mathbf{u})|_p = |f(s)|_p \leq |f|_1$$

for all integers $s \geq 0$. □

Proposition 4.9. There is a positive constant c_4 such that if

$$\log |l(u)|_p \leq -c_4 \left(\left(S_0 + \frac{1}{p-1} + e_L \right) T \log p + S_0 T \log S_0 \right)$$

then

$$\log |(\Delta^t \Psi)(s\mathbf{u}_0)|_p \leq -(\epsilon S_0 - e_L) T \log p$$

for $t \in \mathbb{N}^n$ with $|t| < T$ and for $s \in \mathbb{N}$.

Proof. By Lemma 4.4

$$|(\Delta^t \Psi)(s\mathbf{u}) - (\Delta^t \Psi)(s\mathbf{u}_0)|_p \leq |\delta_L^{-1}|_p^{2nT} |l(u)|_p = p^{2ne_L T} |l(u)|_p,$$

and by Proposition 4.8

$$|(\Delta^t \Psi)(s\mathbf{u})|_p \leq p^{-(\epsilon S_0 - e_L)T} \max \left\{ 1, p^{((2n-1)e_L + \epsilon S_0 + \frac{1}{p-1})T} S_0^{S_0 T} |l(u)|_p \right\}.$$

Hence

$$\begin{aligned} |(\Delta^t \Psi)(s\mathbf{u}_0)|_p &\leq \max \{ |(\Delta^t \Psi)(s\mathbf{u})|_p, |(\Delta^t \Psi)(s\mathbf{u}) - (\Delta^t \Psi)(s\mathbf{u}_0)|_p \} \\ &\leq p^{-(\epsilon S_0 - e_L)T} \max \left\{ 1, p^{((2n-1)e_L + \epsilon S_0 + \frac{1}{p-1})T} S_0^{S_0 T} |l(u)|_p, \right. \\ &\quad \left. p^{((2n-1)e_L + \epsilon S_0)T} |l(u)|_p \right\} \\ &\leq p^{-(\epsilon S_0 - e_L)T} \max \left\{ 1, p^{((2n-1)e_L + \epsilon S_0 + \frac{1}{p-1})T} S_0^{S_0 T} |l(u)|_p \right\}. \end{aligned}$$

On the other hand

$$p^{((2n-1)e_L + \epsilon S_0 + \frac{1}{p-1})T} S_0^{S_0 T} |l(u)|_p \leq 1$$

if and only if

$$|l(u)|_p \leq p^{-((2n-1)e_L + \epsilon S_0 + \frac{1}{p-1})T} S_0^{-S_0 T}.$$

In other words, if

$$\log |l(u)|_p \leq - \left((2n-1)e_L + \epsilon S_0 + \frac{1}{p-1} \right) T \log p - S_0 T \log S_0$$

then

$$|(\Delta^t \Psi)(s\mathbf{u}_0)|_p \leq p^{-(\epsilon S_0 - e_L)T}.$$

This means that there is a positive constant c_4 such that if

$$\log |l(u)|_p \leq -c_4 \left(\left(S_0 + \frac{1}{p-1} + e_L \right) T \log p + S_0 T \log S_0 \right)$$

then

$$\log |(\Delta^t \Psi)(s\mathbf{u}_0)|_p \leq -(\epsilon S_0 - e_L)T \log p.$$

The proposition is proved. \square

4.6. A lower bound. Using the Liouville's inequality, we derive the following result that will be crucial in the proof of the main result.

Proposition 4.10. Let s be an integer such that $0 \leq s < S$. Assume that Ψ has a zero at $s\mathbf{u}_0$ of exact order T' along \mathcal{W} for some positive integer T' . Let t be any element in $\mathbb{Z}_{\geq 0}^n$ with $|t| = T'$ such that $(\Delta^t \Psi)(s\mathbf{u}_0) \neq 0$, then

$$\log |(\Delta^t \Psi)(s\mathbf{u}_0)|_p > -c_5 (T' (h_L + \log \delta_L + \log(D + T' d_L)) + D_0 b + D S^2 h)$$

for some positive constant c_5 .

Proof. As in the proof of Proposition 4.2, for $y \in K_v$ and for $x \in \Lambda_v^n$ we define

$$\Psi_s^*(y, x) := \Psi(y, su+x), \quad E_s(x) := E_0(su, x), \quad \Psi_s(y, x) := \Psi_s^*(y, x) E_s(x)^D.$$

By our assumption

$$0 = (\Delta^\tau \Psi)(s\mathbf{u}_0) = (\Delta^\tau \Psi)(0, su) = (\Delta^\tau \Psi_s^*)(0, 0)$$

for $\tau \in \mathbb{N}^n$ with $|\tau| < T'$. Leibniz' rule gives

$$(\Delta^\tau \Psi_s)(0, 0) = (\Delta^\tau (\Psi_s^* E_s^D))(0, 0) = 0.$$

Using Leibniz' rule again, one gets

$$(\Delta^t \Psi)(s\mathbf{u}_0) = (\Delta^t (\Psi_s E_s^{-D}))(0, 0) = (\Delta^t \Psi_s)(0, 0) E_s^{-D}(0).$$

The same arguments as in the proof of Proposition 4.2 (just replace S_0 by S) show that $h((\Delta^t \Psi_s)(0, 0))$ is

$$\ll T'(h_L + \log \delta_L + \log(D + T' \log d_L)) + D_0 b + DS^2 h.$$

Furthermore

$$h(E_s^{-D}(0)) = h(E_0(su, 0)^{-D}) = Dh(E_0(su, 0)) \ll DS^2 h.$$

Since $(\Delta^t \Psi)(s\mathbf{u}_0) \neq 0$ Liouville's inequality gives

$$\begin{aligned} \log |(\Delta^t \Psi)(s\mathbf{u}_0)|_p &\gg -h((\Delta^t \Psi)(s\mathbf{u}_0)) = -h((\Delta^t \Psi_s)(0, 0)E_s(0)^{-D}) \\ &\gg -(T'(h_L + \log \delta_L + \log(D + T' d_L)) + D_0 b + DS^2 h) \end{aligned}$$

and the proposition follows. \square

4.7. Multiplicity estimates. Another crucial point for proving the theorem is the following lemma. For the proof we use [17], but we also refer to [21] (and to [22], where the multiplicity estimates part has been published); the former mentioned result is a modification of the multiplicity estimate part of latter habilitation thesis.

Lemma 4.11. Let η denote the point $(0, \gamma)$ and $\Gamma(\eta)$ denote the set $\{\eta^i; i \in \mathbb{N}\}$. Let $H(\mathcal{H}; D_0, D)$ and $H(\mathcal{G}; D_0, D)$ be the Hilbert-Samuel functions associated with the ideal of \mathcal{H} and \mathcal{G} respectively. If Ψ vanishes at any point of the set $\{s\mathbf{u}_0; 0 \leq s < S\}$ along \mathcal{W} of order $\geq T$, then there are a connected algebraic subgroup \mathcal{H} defined over K distinct from \mathcal{G} and a positive constant c_6 satisfying that the quantity

$$\left(\frac{T + \text{codim}_{\mathcal{W}_p} \mathcal{W}_p \cap T_{\mathcal{H}}}{\text{codim}_{\mathcal{W}_p} \mathcal{W}_p \cap T_{\mathcal{H}}} \right) \text{card}((\Gamma(\eta) + \mathcal{H})/\mathcal{H}) H(\mathcal{H}; D_0, D)$$

is bounded from above by $c_6 H(\mathcal{G}; D_0, D)$, where $\mathcal{W}_p := \mathcal{W} \otimes_{K_v} \mathbb{C}_p$ and $T_{\mathcal{H}} = \text{Lie}(\mathcal{H}) \otimes_K \mathbb{C}_p$.

Proof. We associate with P the bi-homogeneous polynomial P^h in $N+2$ variables $Y_0, Y_1, X_0, \dots, X_N$ of degree D_0 in Y_0, Y_1 and degree D in X_0, \dots, X_N defined by

$$P^h(Y_0, Y_1, X_0, \dots, X_N) := Y_0^{D_0} P(Y_1/Y_0, X_0, \dots, X_N).$$

Since $\text{ord}_{s\mathbf{u}_0, \mathcal{W}} \Psi \geq T$ the order at any point $s\mathbf{u}_0$ along \mathcal{W} of the analytic function $P^h(1, y, 1, f_1(x), \dots, f_N(x))$ is at least T . This also means that the order of $P^h(1, y, 1, f_1(x), \dots, f_N(x))$ along \mathcal{W}_p at any point $s\mathbf{u}_0$ is at least T . Therefore the lemma follows immediately from Theorem 2.1 of [17]. \square

4.8. Choice of parameters and proof of Theorem 2.1. We choose parameters as follows. Let c be a large enough positive constant and

$$\begin{aligned} S_0 &= [c\omega_L(\log b + \log h)], \\ D_0 &= [c^{5n+1}S_0^{n+1}h^n], \\ S &= [c^2S_0], \\ D &= [c^{5n+1}S_0^n b h^{n-1}], \\ T &= [c^{5n+6}S_0^{n+1}b h^n], \end{aligned}$$

where $[x]$ for real x is defined to be the largest integer less than or equal to x . Our parameters satisfy $D_0 D^n \geq c_2 S_0 T^n$. Proposition 4.2 gives a polynomial P in $N+2$ variables Y, X_0, \dots, X_N with coefficients in \mathcal{O}_K , homogeneous in X_0, \dots, X_N of degree D , and with $\deg P_Y \leq D_0$ such that

1. P does not vanish identically on \mathcal{G} ,
2. $(\Delta^t \Psi)(s\mathbf{u}_0) = 0, \forall 0 \leq s < S_0, \forall t = (t_1, \dots, t_n), 0 \leq t_1, \dots, t_n < 2T$,
3. $h(P) \leq c_3(T(h_L + \log \delta_L + \log(D + Td_L)) + D_0 b + DS_0^2 h)$;

here we write Ψ for Ψ_P .

Lemma 4.12.

$$\log |l(u)|_p > -c_4 \left(\left(S_0 + \frac{1}{p-1} + e_L \right) T \log p + S_0 T \log S_0 \right).$$

Proof. On assuming that

$$\log |l(u)|_p \leq -c_4 \left(\left(S_0 + \frac{1}{p-1} + e_L \right) T \log p + S_0 T \log S_0 \right)$$

Proposition 4.9 gives

$$\log |(\Delta^t \Psi)(s\mathbf{u}_0)|_p \leq -(\epsilon S_0 - e_L) T \log p.$$

We shall show that the order of Ψ along \mathcal{W} at any point of the set $\{s\mathbf{u}_0; 0 \leq s < S\}$ is at least T . Otherwise there is some point $s_0\mathbf{u}_0$ with $0 \leq s_0 < S$ at which the exact order along \mathcal{W} is $T_0 < T$. This means that there exists a n -tuple $\tau \in \mathbb{N}^n$ such that $|\tau| = T_0$ and $(\Delta^\tau \Psi)(s_0\mathbf{u}_0) \neq 0$. We apply Proposition 4.10 to get

$$\log |(\Delta^\tau \Psi)(s_0\mathbf{u}_0)|_p > -c_5(T_0(h_L + \log \delta_L + \log(D + T_0 d_L)) + D_0 b + DS^2 h).$$

The comparison with the lower bound above implies that the quantity

$$-(\epsilon S_0 - e_L) T \log p$$

is bounded from below by

$$-c_5(T(h_L + \log \delta_L + \log(D + Td_L)) + D_0 b + DS^2 h).$$

This implies that

$$(\epsilon S_0 - e_L)T \log p \leq c_5(T(h_L + \log \delta_L + \log(D + Td_L)) + D_0b + DS^2h)$$

and shows that

$$\begin{aligned} & \left(\frac{1}{3d^2} \log 2 \right) T(S_0 - e_L) \\ & \leq c_5(T(h_L + \log \delta_L + \log(D + Td_L)) + D_0b + DS^2h). \end{aligned}$$

This means that there is a positive constant c_7 satisfying

$$T(S_0 - e_L) \leq c_7(T(h_L + \log \delta_L + \log(D + Td_L)) + D_0b + DS^2h).$$

We get a contradiction because this cannot hold if c is sufficiently large. Therefore Ψ vanishes at any point of the set $\{\mathbf{su}_0; 0 \leq s < S\}$ of order at least T along \mathcal{W} . By Lemma 4.11, there is a connected algebraic subgroup \mathcal{H} defined over K distinct from \mathcal{G} satisfying

$$\left(\frac{T + \text{codim}_{\mathcal{W}_p} \mathcal{W}_p \cap T_{\mathcal{H}}}{\text{codim}_{\mathcal{W}_p} \mathcal{W}_p \cap T_{\mathcal{H}}} \right) \text{card}((\Gamma(\eta) + \mathcal{H})/\mathcal{H}) H(\mathcal{H}; D_0, D)$$

is bounded from above by $c_6 H(\mathcal{G}; D_0, D)$. Since G and \mathbb{G}_a are disjoint, there are subgroups H_a of \mathbb{G}_a and H of G (defined over K) such that $\mathcal{H} = H_a \times H$. Let n_a be the dimension of H_a and n' be the dimension of H . We know that $H(\mathcal{H}; D_0, D) \gg D_0^{n_a} D^{n'}$ and $H(\mathcal{G}; D_0, D) \ll D_0 D^n$. The above inequality gives

$$\left(\frac{T + \text{codim}_{\mathcal{W}_p} \mathcal{W}_p \cap T_{\mathcal{H}}}{\text{codim}_{\mathcal{W}_p} \mathcal{W}_p \cap T_{\mathcal{H}}} \right) \text{card}((\Gamma(\eta) + \mathcal{H})/\mathcal{H}) \ll D_0^{1-n_a} D^{n-n'}.$$

We shall show that H must be the trivial group $\{e\}$. Indeed, if not, then we get a proper quotient $\pi : G \rightarrow G/H$ inducing a linear map $\pi_* : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ of Lie algebras which maps the hyperplane W onto the quotient $(W + \mathfrak{h})/\mathfrak{h}$; here \mathfrak{g} and \mathfrak{h} denote the Lie algebra of G and H respectively. Furthermore we have $\tau(G, W) = \frac{n-1}{n}$ and since (G, W) is semistable over $\overline{\mathbb{Q}}$, it is also semistable over K . This gives

$$\begin{aligned} \tau(G, W) & \leq \tau(G/H, \pi_*(W)) \\ & = \frac{\dim(W + \mathfrak{h}) - \dim \mathfrak{h}}{\dim G - \dim H} \\ & = \frac{\dim(W + \mathfrak{h}) - n'}{n - n'}. \end{aligned}$$

But

$$n - 1 = \dim W \leq \dim(W + \mathfrak{h}) \leq n$$

and this shows that $\dim(W + \mathfrak{h})$ must be n , i.e. $\dim(\mathcal{W}_p + T_{\mathcal{H}}) = n$. This gives

$$\text{codim}_{\mathcal{W}_p} \mathcal{W}_p \cap T_{\mathcal{H}} = \dim(\mathcal{W}_p + T_{\mathcal{H}}) - \dim T_{\mathcal{H}} = n + 1 - n_a - n',$$

and shows that

$$\binom{T+n+1-n_a-n'}{n+1-n_a-n'} \ll D_0^{1-n_a} D^{n-n'}.$$

We deduce that

$$T^{n+1-n_a-n'} \leq c_8 D_0^{1-n_a} D^{n-n'},$$

for some positive constant c_8 and get a contradiction to $T > cD_0, cD$. As a consequence we obtain $H = \{e\}$, and therefore $T_{\mathcal{H}} \cap \mathcal{W}_p$ must be trivial.

One gets

$$\text{codim}_{\mathcal{W}_p} \mathcal{W}_p \cap T_{\mathcal{H}} = \dim \mathcal{W}_p = n.$$

Moreover, $\Gamma(\gamma) \cap \mathcal{H}$ must also be trivial and hence

$$\text{card}((\Gamma(\gamma) + \mathcal{H})/\mathcal{H}) = \text{card} \Gamma(\eta) = S.$$

We obtain

$$\binom{T+n}{n} S \ll D_0^{1-n_a} D^n \leq D_0 D^n.$$

This therefore shows that $T^n S \leq c_9 D_0 D^n$ for some positive constant c_9 , and again gives a contradiction because of the choice of parameters. The lemma is proved. \square

In order to finish the proof of the theorem, we use the above lemma and the fact that $\log r_p^{-1} = \frac{\log p}{p-1} < 2$ to get

$$\begin{aligned} \log |l(u)|_p &> -c_{10}(S_0 T \log p + S_0 T \log S_0 + T e_L \log p) \\ &> -c_{11}(S_0^{n+2} b h^n \log p + S_0^{n+2} (\log S_0) b h^n) \\ &> -c_{12} S_0^{n+3} b h^n \log p \end{aligned}$$

for some positive constants c_{10}, c_{11} and c_{12} . In other words there is a positive constant c_0 independent of b, h, p such that

$$\log |l(u)|_p > -c_0 \omega_L^{n+3} b h^n (\log b + \log h)^{n+3} \log p.$$

The first assertion of the theorem is therefore proved and this together with Section 2.2 completes the proof of the theorem.

5. ACKNOWLEDGMENTS

The authors are most grateful to Professor G. Wüstholz for very insightful discussions and for constant encouragement to work on the topic. Moreover, the authors thank an anonymous referee for his careful reading of the manuscript. The first author was supported by Austrian Science Fund (FWF): P24574. The second author was supported by grant PDFMP2_122850 funded by the Swiss National Science Foundation (SNSF).

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